# Berkeley Math Circle
# Monthly Contest 7 – Solutions

1. Define a "word" to be a string of at most ten letters taken from the English alphabet. (The letters do not have to be distinct.) Prove that the number of "words" is divisible by 27.

   *Solution.* The total number of 9- and 10- letter words is divisible by 27, because they can be partitioned into groups of 27 with each group containing a 9-letter word and the twenty-six 10-letter words formed by adding a letter at the end of it.

   For the same reason, the number of 7- and 8-letter words is divisible by 27; likewise for 5- and 6-, 3- and 4-, 1- and 2-letter words. Therefore the total number of words is divisible by 27.

2. A pool can be filled through four different pipes. If the first and second pipes are operating, it takes 2 hours to fill the pool. With the second and third pipes it takes 3 hours, and with the third and fourth it takes 4 hours. How long does it take to fill the pool if the first and fourth pipes are operating?

   *Solution.* Let $a$, $b$, $c$, $d$ denote the portion of the pool the pipes fill per hour (with $a$ corresponding to the first pipe). Because the first and second pipes fill $\frac{1}{2}$ of the pool per hour, we have that

   $$a + b = \frac{1}{2}.$$

   Analogously, we derive

   $$b + c = \frac{1}{3}$$

   and

   $$c + d = \frac{1}{4}.$$

   But now

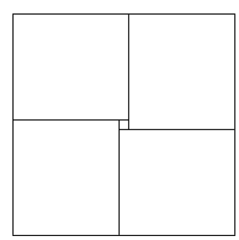
   $$a + d = (a + b) + (c + d) - (b + c) = \frac{1}{2} + \frac{1}{4} - \frac{1}{3} = \frac{5}{12}$$

   so together the first and fourth pipes fill $\frac{5}{12}$ of the pool per hour. Hence they fill the entire pull in $\frac{12}{5}$ hours (which may also be written as 2.4 hours, or 2 hours and 24 minutes).

3. Some soldiers are standing in a line in the east-west direction, each of them facing north. Their officer commands, "Right face!" They should now all be facing east, but, as they are at the very beginning of their military career, some of them get the order wrong and turn to the west. Every soldier who is then facing his neighbor immediately concludes that he has made a mistake and performs a $180°$ turn within 1 second. The process continues, so that any two soldiers who are now facing each other perform $180°$ turns within 1 second. Prove that after some time, the soldiers will stop moving.

   *Solution.* Give the soldiers ID numbers from 1 upwards from west to east. At any moment, define the *confusion index* to be the sum of the ID numbers of the soldiers who are facing west. Note that whenever a pair of soldiers turn around, soldier $n + 1$ turns from west to east (decreasing the confusion index by $n + 1$) while soldier $n$ turns from east to west (increasing the confusion index by $n$). So overall, the confusion index decreases by 1 for every pair of turns. Since the confusion index is a nonnegative integer, it cannot keep decreasing forever.

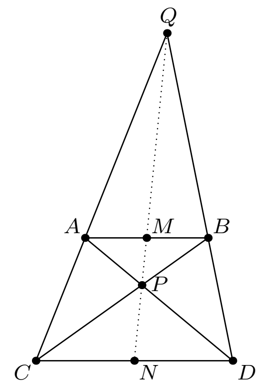
4. A $23 \times 23$ square is divided into smaller squares of dimensions $1 \times 1$, $2 \times 2$, and $3 \times 3$. What is the minimum possible number of $1 \times 1$ squares?

   *Solution.* Color the rows of the square black and white alternately, so the top and bottom rows are black. Then each $2 \times 2$ tile covers two cells of each color, and each $3 \times 3$ tile covers six of one color and three of the other. In particular, if only $2 \times 2$ and $3 \times 3$ tiles are used, the difference between the number of black and white cells covered is divisible by 3. But the entire board has 23 more black than white cells (if the bottom row were removed, the colors would be equally represented). So there must be at least one $1 \times 1$ tile.
   To construct the required tiling with only one $1 \times 1$ tile, first use $3 \times 3$ tiles to build a $9 \times 12$ rectangle and $2 \times 2$ tiles to build a $2 \times 12$ rectangle. Join these two rectangles to form an $11 \times 12$ rectangle. Then use four copies of this $11 \times 12$ rectangle, together with the $1 \times 1$ square, to build a $23 \times 23$ square as shown in the diagram.

5. Let $ABCD$ be a trapezoid with $AB \parallel CD$. Let $M$ and $N$ be the respective midpoints of $AB$ and $CD$. Diagonals $AC$ and $BD$ meet at $P$, and lines $AD$ and $BC$ meet at $Q$. Prove that $M$, $N$, $P$, and $Q$ are collinear.

*Solution.* Note that $\triangle QAB \sim \triangle QDC$ (by corresponding angles). Since $QM$ and $QN$ are corresponding medians, we have $\angle AQM = \angle DQN$ and hence $Q$, $M$, and $N$ are collinear.




Similarly, note that $\triangle PAB \sim \triangle PCD$. Since $PM$ and $PN$ are corresponding medians, we have $\angle APM = \angle CPN$ and hence $P$, $M$, and $N$ are collinear. Since $P$ and $Q$ lie on line $MN$, all four of these points are collinear.

6. Let $a_1 = 5$ and $a_{n+1} = a_n^3 - 2a_n^2 + 2$ for all $n \geq 1$. Prove that if $p$ is a prime divisor of $a_{2014} + 1$ and $p \equiv 3 \bmod 4$, then $p = 3$.

   *Solution.* Observe that $a_{n+1} - 2 = a_n^2(a_n - 2)$ for all $n \geq 1$. By induction on $n$ we obtain

$$a_{n+1} - 2 = 3a_n^2 a_{n-1}^2 \cdots a_1^2$$

   for all $n \geq 1$. Therefore

$$a_{2014} + 1 = 3(a_{2013}^2 a_{2012}^2 \cdots a_1^2 + 1) = 3[(a_{2013}a_{2012} \cdots a_1)^2 + 1].$$

   It is well known that if $q$ is a prime divisor of $x^2 + 1$ ($x \in \mathbb{Z}$) then $q \not\equiv 3 \bmod 4$. (This is a special case of a result known as *quadratic reciprocity*.) So the $p$ in the problem must be 3.

7. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) - f(x) = (x + f(y))^4 - x^4$$

   for all $x, y \in \mathbb{R}$.

   *Solution.* The solutions are $f(x) = 0$ and $f(x) = x^4 + k$ for any real constant $k$. These clearly satisfy the equation.

   Assume $f$ is not the 0 function and pick any $y$ such that $f(y) \neq 0$. Then $(x + f(y))^4 - x^4$ is a cubic polynomial in $x$. In particular, it attains every real value. So every real number has the form $f(x + f(y)) - f(x)$ and in particular the form $f(a) - f(b)$ for some $a$ and $b$.

   Substituting $(-f(b), a)$ and $(-f(b), b)$ into the equation,

$$f(f(a) - f(b)) - f(-f(b)) = (f(a) - f(b))^4 - f(b)^4$$
$$f(0) - f(-f(b)) = -f(b)^4$$

   and subtracting,

$$f(f(a) - f(b)) - f(0) = (f(a) - f(b))^4.$$

   But $f(a) - f(b)$ can equal an arbitrary real number $x$, so $f(x) = x^4 + f(0)$ as desired.