# Berkeley Math Circle
# Monthly Contest 7 – Solutions

1. Which is larger,

$$3^{3^{3^3}} \quad \text{or} \quad 2^{2^{2^2}}\,?$$

*Remark.* Note that $3^{3^3}$ means $3^{(3^3)}$, not $(3^3)^3$ and so on.

*Solution.* Starting to evaluate, the first expression becomes

$$A = 3^{3^{3^3}} = 3^{3^{27}}.$$

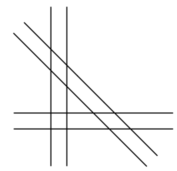
The second expression is

$$B = 2^{2^{2^2}} = 2^{2^4} = 2^{2^{16}}.$$

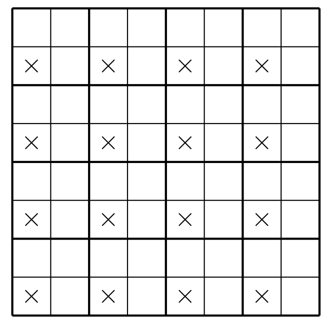
It is now obvious that

$$A = 3^{3^{27}} > 3^{3^{16}} > 3^{2^{16}} > 2^{2^{16}} = B$$

so the first expression is the larger one.



Problem 2



Problem 3

2. In the plane, six lines are drawn such that such that no three of them meet at one point. Can it happen that there are exactly (a) 12 (b) 16 intersection points where two lines meet?

*Solution.* (a) The answer is yes. Some pairs of lines must be parallel. One of several solutions is shown.

(b) The answer is no. Label the lines $a$, $b$, $c$, $d$, $e$, $f$. Since two lines cannot meet at more than one point, each intersection point comes from a unique pair of lines:

$$ab,\ ac,\ ad,\ ae,\ af,\ bc,\ bd,\ be,\ bf,\ cd,\ ce,\ cf,\ de,\ df,\ ef.$$

There are only 15 pairs. Thus there cannot be 16 points of intersection.

3. What is the maximum number of squares on an $8 \times 8$ chessboard on which pieces may be placed so that no two of these squares touch horizontally, vertically, or diagonally?

*Solution.* The answer is 16.

Divide the chessboard into sixteen $2 \times 2$ regions (see figure). Notice that no two pieces can lie in the same region. So there are at most 16 pieces. By placing a piece in the lower left corner of each region, we see that 16 is achievable.

4. $n$ boxes initially contain $1, 2, \ldots, n$ marbles respectively ($n \geq 1$). Charlotte first adds a marble to each box. Then she adds a marble to each box in which the number of marbles is divisible by 2, then a marble to each box in which the number of marbles is divisible by 3, and so on, until she adds a marble to each box in which the number of marbles is divisible by $n$. For which values of $n$ does the procedure end with exactly $n + 1$ marbles in every box?

*Solution.* The answer is: all $n$ such that $n + 1$ is prime.

*Lemma* 1. *If box $A$ is to the left of box $B$ (we can arrange the boxes so that they initially contain $1, \ldots, n$ marbles respectively from left to right), then at no stage of the process can box $A$ have more marbles than box $B$.*

*Proof.* At the outset box $A$ certainly has fewer marbles than box $B$. Each subsequent stage adds at most one marble to each box, with one of two outcomes: (1) $A$ still has fewer marbles than $B$; (2) $A$ and $B$ have the same number of marbles, at which point the rules force $A$ and $B$ to have the same number of marbles throughout the process. This proves the lemma. □

In view of this lemma, for all the boxes to end up with $n + 1$ marbles, it is necessary and sufficient for the first box and the last box to end up with $n + 1$ marbles. The first box will certainly have $n + 1$ marbles because Charlotte adds a marble to it on every turn. The last box, which begins with $n$ marbles, will have $n+1$ after the first turn. If $n+1$ is prime, then since it is not divisible by any of the numbers $2, 3, 4, \ldots, n$, the box will not pick up any more marbles and thus will end up with $n+1$. Then the intervening boxes, being sandwiched between two boxes with $n+1$ marbles, will also have $n+1$ marbles.

If $n+1$ is composite, then it is divisible by one of the numbers $2, 3, 4, \ldots, n$. Hence the box will pick up another marble and will finish with at least $n+2$ marbles.

5. In a bag are $n$ fair, six-sided dice whose faces are colored white and red in such a way that the total numbers of white and red sides are equal. Let $p$ be the probability that the same color comes up twice when taking one die randomly out of the bag and throwing it twice. Let $q$ be the probability that the same color comes up twice when taking two dice randomly out of the bag and throwing them at the same time. Prove that

$$p + (n-1)q = \frac{n}{2}.$$

*Solution.* Consider the following procedure: Remove one die randomly from the bag, roll it, replace it in the bag, remove another die randomly from the bag, and roll it. It is clear that each roll is an independent and random choice of one of the $6n$ sides of all the dice; hence the probability of getting the same color twice is $1/2$. On the other hand, we can decompose the probability as follows:

- With probability $1/n$, we will pick the same die twice. Then the probability that the same color comes up is $p$.
- With probability $(n-1)/n$, we will *not* pick the same die twice. Then the two dice we roll form a random pair of distinct dice, and the probability that they will come up the same color is $q$.

Adding up the probabilities, we conclude that

$$\frac{1}{n} \cdot p + \frac{n-1}{n} \cdot q = \frac{1}{2},$$

that is,

$$p + (n-1)q = \frac{n}{2}.$$

6. Prove that there exist a point $A$ on the graph of $f(x) = x^4$ and a point $B$ on the graph of $g(x) = x^4 + x^2 + x + 1$ such that the distance between $A$ and $B$ is less than $1/100$.

*Solution.* We will find two such points that lie on the same horizontal line. Write $\epsilon = 1/100$ for brevity. Let $u > 1$ be a large real number to be determined later, and let $B$ be the point $(u, g(u))$. Clearly $f(u) < g(u)$. On the other hand, we claim that $f(u+\epsilon) > g(u)$ if $u$ is large enough. This is because $f(u+\epsilon) - g(u)$ is a cubic polynomial with positive leading coefficient: explicitly,

$$\begin{aligned} f(u+\epsilon) - g(u) &> u^4 + 4u^3\epsilon - g(u) \\ &= u^4 + 4u^3\epsilon - u^4 - u^2 - u - 1 \\ &= 4u^3\epsilon - u^2 - u - 1 \\ &> 4u^3\epsilon - 3u^2 \\ &> 4u^2(u\epsilon - 1) \end{aligned}$$

which is positive if $u > 100$. Now, since $f(u) < g(u) < f(u+\epsilon)$, there is a $v$, $u < v < u+\epsilon$ with $f(v) = g(u)$ (by the Intermediate Value Theorem). Then $A = (v, f(v))$ lies at a distance less than $\epsilon$ from $B = (u, g(u))$.

7. Let $p$ be a prime number, and $f(x_1, \ldots, x_n)$ be a polynomial with integer coefficients of total degree less than $n$. Prove that the number of ordered $n$-tuples $(x_1, \ldots, x_n)$ with $0 \le x_i < p$ such that $f(x_1, \ldots, x_n)$ is an integer multiple of $p$ is an integer multiple of $p$.

*Solution.* By Fermat's little theorem, the expression

$$f(x_1, \ldots, x_n)^{p-1}$$

equals 0 mod $p$ if $f(x_1, \ldots, x_n)$ is 0 mod $p$ and 1 mod $p$ otherwise. So

$$S = \sum_{0 \le x_1, \ldots, x_n < p} f(x_1, \ldots, x_n)^{p-1} \equiv p^n - z \equiv -z,$$

where $z$ is the number of zeros of $f$ mod $p$. To prove $z \equiv 0$, it suffices to prove that $S \equiv 0$. Expanding $f^{p-1}$ as a polynomial, we see that $S$ is a sum of terms of the form

$$\sum_{0 \le x_1, \dots, x_n < p} c_{a_1, \dots, a_n} x_1^{a_1} \cdots x_n^{a_n} = c_{a_1, \dots, a_n} \Big( \sum_{0 \le x_1 < p} x_1^{a_1} \Big) \cdots \Big( \sum_{0 \le x_n < p} x_n^{a_n} \Big).$$

We claim that one of the factors in parentheses is zero mod $p$. We know $f$ has total degree at most $n-1$, so $f^{p-1}$ has total degree at most $(n-1)(p-1)$, and hence some $a_i$ is less than $p-1$. So it suffices to prove that

$$T := \sum_{0 \le x < p} x^a \equiv 0 \mod p$$

for $0 \le a < p-1$. For $a = 0$ it is obvious (note that we are using the convention $0^0 = 1$). Otherwise, note that the equation $y^a \equiv 1$ has at most $a$ roots mod $p$, so there is a $y$ such that $y^a \not\equiv 1$. Now

$$(y^a - 1)T = \sum_{0 \le x < p} (xy)^a - \sum_{0 \le x < p} x^a,$$

which is 0 mod $p$ since the terms of the first sum are just a permutation of those of the second. Consequently $T \equiv 0$ mod $p$.