Berkeley Math Circle
Monthly Contest 7
Due April 8, 2014

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 7, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Which is larger,
$$3^{3^{3^3}} \quad \text{or} \quad 2^{2^{2^{2^2}}}?$$
Remark. Note that 3^{3^3} means $3^{(3^3)}$, not $(3^3)^3$ and so on.

2. In the plane, six lines are drawn such that such that no three of them meet at one point. Can it happen that there are exactly (a) 12 (b) 16 intersection points where two lines meet?

3. What is the maximum number of squares on an 8×8 chessboard on which pieces may be placed so that no two pieces lie on squares that touch horizontally, vertically, or diagonally?

4. n boxes initially contain $1, 2, \dots, n$ marbles respectively ($n \geq 1$). Charlotte first adds a marble to each box. Then she adds a marble to each box in which the number of marbles is divisible by 2, then a marble to each box in which the number of marbles is divisible by 3, and so on, until she adds a marble to each box in which the number of marbles is divisible by n. For which values of n does the procedure end with exactly $n + 1$ marbles in every box?

5. In a bag are n fair, six-sided dice whose faces are colored white and red in such a way that the total numbers of white and red sides are equal. Let p be the probability that the same color comes up twice when taking one die randomly out of the bag and throwing it twice. Let q be the probability that the same color comes up twice when taking two dice randomly out of the bag and throwing them at the same time. Prove that
$$p + (n-1)q = \frac{n}{2}.$$

6. Prove that there exist a point A on the graph of $f(x) = x^4$ and a point B on the graph of $g(x) = x^4 + x^2 + x + 1$ such that the distance between A and B is less than $1/100$.

7. Let p be a prime number, and $f(x_1, \dots, x_n)$ be a polynomial with integer coefficients of total degree less than n. Prove that the number of ordered n-tuples $(x_1, \dots, x_n)$ of integers with $0 \leq x_i < p$ such that $f(x_1, \dots, x_n)$ is an integer multiple of p is an integer multiple of p.