Berkeley Math Circle
Monthly Contest 6 – Solutions

1. Six distinct numbers are chosen from the list $1, 2, \ldots, 10$. Prove that their product is divisible by a perfect square greater than 1.

   Solution 1. If all the odd numbers are chosen, then in particular 9 is chosen and the product is divisible by 9.

   If not all the odd numbers are chosen, then at most 4 odds and thus at least 2 evens are chosen. Therefore the product is divisible by 4.

   Solution 2. At most one of the numbers is 1, and each of the other numbers has at least one prime factor. Therefore the product of the six chosen numbers consists of at least *five* prime factors. But there are only *four* prime factors in the numbers from 1 to 10: 2, 3, 5, and 7. So some prime appears twice, making a square divisor greater than 1.

2. The sum of the digits of all counting numbers less than 13 is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 1 + 0 + 1 + 1 + 1 + 2 = 51.$$

   Find the sum of the digits of all counting numbers less than 1000.

   Solution. The counting numbers less than 1000 are simply the three-digit numbers, as long as we allow numbers to start with 0 (except the number 000, but including this does not affect the digit sum).

   Each of the ten digits appears as a hundreds digit 100 times, because it can be paired with any of the 10 tens digits and the 10 units digits. So the sum of the 1000 hundreds digits is

$$100(0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 100 \cdot 45 = 4500.$$

   Similarly, every digit appears as a tens digit 100 times and as a units digit 100 times, so the sums of the tens digits and of the units digits are also 4500. Consequently, the sum of all the digits is $3 \cdot 4500 = 13500$.

3. Let a, b, c, d, e, and f be decimal digits such that the six-digit number $\overline{abcdef}$ is divisible by 7. Prove that the six-digit number $\overline{bcdefa}$ is divisible by 7.

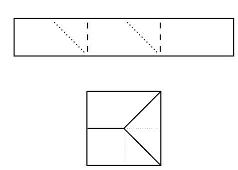
   Remark. In this problem, it is permissible for one or both of the numbers to begin with the digit 0.

   Solution. We have

$$\begin{aligned}
\overline{bcdefa} &= \overline{abcdefa} - \overline{a000000} \\
&= \overline{abcdef0} - \overline{a000000} + a \\
&= 10 \cdot \overline{abcdef} - 1000000 \cdot a + a \\
&= 10 \cdot \overline{abcdef} - 999999 \cdot a \\
&= 10 \cdot \overline{abcdef} - 7 \cdot 142857 \cdot a.
\end{aligned}$$

   The first term is a multiple of 7 because $\overline{abcdef}$ is, and the second clearly has 7 as a factor. So their difference $\overline{abcdef}$ is divisible by 7.

4. The diagram shows how a 1×6 sheet of paper can be folded into the shape of a 2×2 square (dotted and dashed lines represent mountain and valley folds respectively). Can a 5×5 sheet of paper be folded into the shape of



   (a) a 1×8 rectangle?

   (b) a 1×7 rectangle?

   Remark. We assume of course that the paper is infinitely thin, and the creases must be finitely many straight line segments.

   Solution. (a) The answer is no.

   Because the shortest path between two points is a straight line, two points in the final shape cannot be farther apart than they were in the initial sheet of paper. But two opposite corners of the 1×8 rectangle are a distance of $\sqrt{1^2 + 8^2} = \sqrt{65}$ apart, while the longest distance within the 5×5 sheet is found between opposite corners and is $\sqrt{5^2 + 5^2} = \sqrt{50}$. Therefore the folding is impossible.

Remark. The distance has been kept as $\sqrt{50}$ instead of simplified to $5\sqrt{2}$ to make clear that it is less than $\sqrt{65}$.

(b) The answer is again no.

Two opposite corners of the 1×7 rectangle lie at a distance of exactly $\sqrt{1^2+7^2} = \sqrt{50}$. Therefore they cannot come from any points of the 5×5 sheet except opposite corners. Similarly, the other two corners of the 1×7 rectangle come from the other two corners of the 5×5. But now we have two adjacent corners of the 5×5 sheet, which are a distance 5 apart, mapping to the endpoints of the edge of length 7 in the 1×7 rectangle, which is impossible.

5. Define a function f on the real numbers by

$$f(x) = \begin{cases} 2x & \text{if } x < 1/2 \\ 2x-1 & \text{if } x \geq 1/2. \end{cases}$$

Determine all values x satisfying $f(f(f(f(f(x))))) = x$.

Solution. The answer is the 32 values $0, \frac{1}{31}, \frac{2}{31}, \ldots, \frac{30}{31}, 1$.

If $x < 0$, then $f(x) = 2x < x$ so the sequence $x, f(x), f(f(x)), \ldots$ is strictly decreasing and cannot return to x.

If $x > 1$, similarly $f(x) = 2x - 1 > x$ so the sequence $x, f(x), f(f(x)), \ldots$ is strictly increasing and cannot return to x.

If $x = 1$, then $f(x) = 1$ and we have 1 as a solution.

Finally, we assume that $0 \leq x < 1$, so $0 \leq f(x) < 1$ as well. For simplicity let $x_0 = x$ and $x_{n+1} = f(x_i)$ so the equation we are trying to solve is $x_5 = x_0$. Note that $2x_n - x_{n+1}$ is an integer (either 0 or 1) for each n, so

$$32x_0 - x_5 = 16(2x_0 - x_1) + 8(2x_1 - x_2) + 4(2x_2 - x_3) + 2(2x_3 - x_4) + (2x_4 - x_5)$$

must be an integer as well. If we assume $x_5 = x_0$ we deduce that $31x_0$ is an integer. Conversely, if $31x_0$ is an integer, then $x_5 - x_0$ is an integer and this integer must be 0 because $0 \leq x_0, x_5 < 1$. Thus the solutions in this range are exactly the multiples of $\frac{1}{31}$: $0, \frac{1}{31}, \frac{2}{31}, \ldots, \frac{30}{31}$.

6. Vandal Evan cut a rectangular portrait of Professor Zvezda along a straight line. Then he cut one of the pieces along a straight line, and so on. After he had made 100 cuts, Professor Zvezda walked in and forced him to pay 2 cents for each triangular piece and 1 cent for each quadrilateral piece. Prove that Vandal Evan paid more than \$1.

Solution. First note that the total number of sides increases by at most 4 at each cut. This is because two new sides are created along the cut, and the two endpoints of the cut may optionally divide other sides into two parts. Therefore, since there are initially 4 sides, at the end there are at most 404 sides.

Now note that the cost of a piece (2, 1, or 0 cents as specified in the problem) is at least five minus its number of sides. So the total of the costs of all 101 pieces is at least 505 minus the total number of sides, or at least $505 - 404 = 101$ cents as desired.

7. Let k be a positive integer. Prove that there exist POSITIVE integers $a_0, \ldots, a_k$ such that for all integers $x \geq 0$,

$$x^k = a_0\binom{x}{k} + a_1\binom{x+1}{k} + \cdots + a_{k-1}\binom{x+k-1}{k}.$$

Remark. We use the convention that $\binom{n}{k} = 0$ whenever $k > n \geq 0$.

Solution. Given k and x, let A be the set of all sets of the form

$$\left\{m_1 + \frac{1}{k}, m_2 + \frac{2}{k}, \ldots, m_k + \frac{k}{k}\right\}$$

where the m_i are integers, $1 \leq m_i \leq x$. Clearly, A consists of x^k sets, each of which has k distinct elements. Now we count the elements of A in another way.

Arrange the elements of a member of A in increasing order:

$$n_1 + \frac{p_1}{k} < n_2 + \frac{p_2}{k} < \cdots < n_k + \frac{p_k}{k}. \quad (1)$$

For each permutation $p = (p_1, \ldots, p_k)$ of the numbers $(1, \ldots, k)$, let $A_p \subseteq A$ be the subset consisting of sets whose sorted presentation (1) displays the given sequence p of numerators. We claim that the size of A_p is a binomial coefficient $\binom{x+r_p}{k}$, where r_p depends on p (hence on k) but not on x.

For $n_1, \ldots, n_k$ $(1 \le n_i \le x)$ to define an element of A_p, it is necessary and sufficient that the inequalities (1) hold. So we must have

$$1 \le n_1 \le n_2 \le \cdots \le n_k \le x.$$

Moreover, the strict inequality $n_i < n_{i+1}$ is required whenever $p_i > p_{i+1}$. To condense all these conditions, it is convenient to define nonnegative integers $e_1, e_2, \ldots, e_k$ by

$$e_1 = 0$$
$$e_{i+1} = \begin{cases} e_i + 1 & \text{if } p_i < p_{i+1} \\ e_i & \text{if } p_i > p_{i+1}. \end{cases}$$

Then the conditions may be written as

$$1 \le n_1 + e_1 < n_2 + e_2 < \cdots < n_k + e_k \le x + e_k.$$

We observe that $\{(n_1 + e_1, \ldots, n_k + e_k)\}$ may be any k-element subset of $\{1, 2, \ldots, x + e_k\}$, written in increasing order. So, writing $r_p = e_k$,

$$|A_p| = \binom{x + r_p}{k}.$$

Observe that $0 \le r_p \le k - 1$. Letting a_i be the number of p for which $r_p = i$, we obtain

$$x^k = |A| = \sum_p |A_p| = \sum_{i=0}^{k-1} a_i \binom{x+i}{k}.$$

Here the a_i are clearly nonnegative integers. To prove that they are positive, it suffices to exhibit, for each i, a p with $r_p = i$; the permutation $(n, n-1, \ldots, i+1, 1, 2, \ldots, i)$ is readily seen to work.