Berkeley Math Circle
Monthly Contest 6
Due March 4, 2014

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 5, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Six distinct numbers are chosen from the list $1, 2, \ldots, 10$. Prove that their product is divisible by a perfect square greater than 1.

2. The sum of the digits of all counting numbers less than 13 is

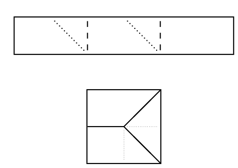
$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 1 + 0 + 1 + 1 + 1 + 2 = 51.$$

   Find the sum of the digits of all counting numbers less than 1000.

3. Let a, b, c, d, e, and f be decimal digits such that the six-digit number $\overline{abcdef}$ is divisible by 7. Prove that the six-digit number $\overline{bcdefa}$ is divisible by 7.

   Remark. In this problem, it is permissible for one or both of the numbers to begin with the digit 0.

4. The diagram shows how a 1×6 sheet of paper can be folded into the shape of a 2×2 square (dotted and dashed lines represent mountain and valley folds respectively). Can a 5×5 sheet of paper be folded into the shape of



   (a) a 1×8 rectangle?

   (b) a 1×7 rectangle?

   Remark. We assume of course that the paper is infinitely thin, and the creases must be finitely many straight line segments.

5. Define a function f on the real numbers by

$$f(x) = \begin{cases} 2x & \text{if } x < 1/2 \\ 2x - 1 & \text{if } x \geq 1/2. \end{cases}$$

   Determine all values x satisfying $f(f(f(f(x)))) = x$.

6. Vandal Evan cut a rectangular portrait of Professor Zvezda along a straight line. Then he cut one of the pieces along a straight line, and so on. After he had made 100 cuts, Professor Zvezda walked in and forced him to pay 2 cents for each triangular piece and 1 cent for each quadrilateral piece. Prove that Vandal Evan paid more than $1.

7. Let k be a positive integer. Prove that there exist POSITIVE integers $a_0, \dots, a_k$ such that for all integers $x \geq 0$,

$$x^k = a_0 \binom{x}{k} + a_1 \binom{x+1}{k} + \cdots + a_k \binom{x+k}{k}.$$

   Remark. We use the convention that $\binom{n}{k} = 0$ whenever $k > n \geq 0$.