Berkeley Math Circle
Monthly Contest 5
Due February 4, 2014

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 5, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Is there a positive integer k such that
$$(\cdots((4\underbrace{!)!)!\cdots)!}_{k} > (\cdots((3\underbrace{!)!)!\cdots)!}_{k+1}?$$

2. Two regular polygons are said to be *matching* if the double of the interior angle of one of them equals the triple of the exterior angle of the other. Find all pairs of matching polygons.

3. A natural number n is chosen between two consecutive square numbers. The smaller square is obtained by subtracting k from n, and the larger one is obtained by adding ℓ to n. Prove that the number $n - k\ell$ is the square of an integer.

4. A positive integer is written in each cell of an 8×8 table so that each entry is the arithmetic mean of some two of its neighbors. Find the maximum number of distinct integers that may appear in the table.

   Remark. The *neighbors* of a cell are the four, three, or two cells with which it shares a side.

5. Prove that
$$-\frac{1}{2} \leq \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)} \leq \frac{1}{2}$$
for all real numbers x and y.

6. Let $ABCDEF$ be a hexagon circumscribing a circle ω. The sides AB, BC, CD, DE, EF, FA touch ω at U, V, W, X, Y, and Z respectively; moreover, U, W, and Y are the midpoints of sides AB, CD, and EF, respectively. Prove that UX, VY, and WZ are concurrent.

7. Let a and b be positive integers. Define a sequence $x_0, x_1, x_2, \ldots$ by $x_0 = 0$, $x_1 = 1$, and $x_{n+2} = ax_{n+1} + bx_n$ for $n \geq 0$. Prove that
$$\frac{x_{m+1}x_{m+2}\cdots x_{m+n}}{x_1x_2\cdots x_n}$$
is an integer for all positive integers m and n.