# Berkeley Math Circle
# Monthly Contest 4 – Solutions

1. A triangle, two of whose sides are 3 and 4, is inscribed in a circle. Find the minimal possible radius of the circle.

   *Solution.* Since the circle has a chord of length 4, its diameter is at least 4 and so its radius is at least 2. To achieve equality, choose a right triangle with hypotenuse 4 and one leg 3 (the other leg will, by the Pythagorean theorem, have length $\sqrt{7}$). Then the midpoint of the hypotenuse is the center of a circle of radius 2 passing through all three vertices.

2. Determine, with proof, whether or not there exist positive integers $a$, $b$, and $c$ such that

$$ab + bc = ac \quad \text{and} \quad abc = 10!.$$

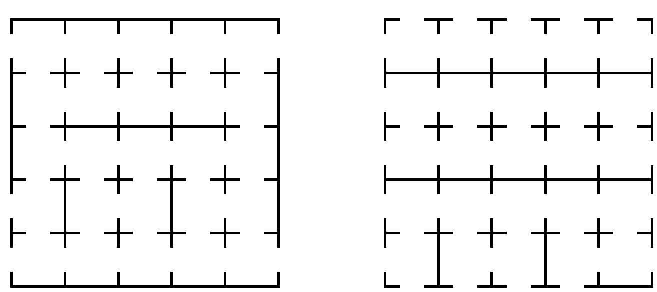
   *Remark.* 10! denotes the factorial $1 \cdot 2 \cdot 3 \cdots 10$.

   *Solution.* The answer is no. Note that 10! has exactly one prime factor of 7. Therefore, exactly one of $a$, $b$, and $c$ is divisible by 7. If 7 divides $b$ (we write this as $7|b$), then $7|ab + bc$ but $7 \nmid ac$, so the equation cannot hold. Likewise, if $7|a$, then $7|ab$, $7|ac$ but $7 \nmid bc$, and if $7|c$, then $7|bc$, $7|ac$ but $7 \nmid ab$. So there are no solutions.

3. A building has the plan of a $5 \times 5$ grid of rooms, each of which has a door in each of its four walls: thus there are 20 doors leading to the outside. The doors are to be opened and closed so that every room has exactly 3 open doors leading from it. Determine the minimum and maximum number of doors to the outside that may be left open.

   *Solution.* If we consider the building as possessing 26 "spaces," namely the outside and the 25 rooms, then each door belongs to two spaces; hence the total of the numbers of open doors to each space must be even. Each room has three open doors, an odd number; hence the number of open doors to the outside must also be odd.

   Also, each of the four corner rooms clearly demands an outside door to be opened. So there are at least 4 open outside doors; coupling this with the parity observation yields at least 5 and at most 19 open outside doors. The diagrams show that both of these extremes are achievable.



4. Prove that for each $n \geq 1$, there is a number $N$ having $n$ digits, each of which is either 1 or 2, such that $N$ is divisible by $2^n$.

   *Solution.* We proceed by induction. For $n = 1$, the number $N = 2$ works. Suppose $N$ works for a given $n$. Consider the two $(n+1)$-digit numbers

$$N_1 = 10^n + N \quad \text{and} \quad N_2 = 2 \cdot 10^n + N.$$

   formed by attaching a single 1 or 2 to the left-hand side of $N$. Note that $N_1$ and $N_2$ are both divisible by $2^n$ (since $N$ and $10^n$ are) and their difference, $N_2 - N_1 = 10^n = 2^n \cdot 5^n$, is divisible by $2^n$ but not by $2^{n+1}$. Therefore, one of $N_1$ and $N_2$ is divisible by $2^{n+1}$, which completes the induction.

   *Remark.* With a bit more analysis, one can prove that the desired number $N$ is unique for each $n$.

5. Let $n \geq 1$ be an integer. How many ways can the rectangle having vertices $(0,0)$, $(n,0)$, $(n,1)$, $(0,1)$ be dissected into $2n$ triangles, all vertices of which have integer coordinates?

   *Remark.* The triangles are considered as positioned on the coordinate plane; in particular, tilings related by rotation and reflection are considered distinct.

   *Solution.* Since the vertices of the tiles must lie within the rectangle and cannot be collinear, all tiles have the form

$$(a,0)(b,0)(c,1) \quad \text{or} \quad (a,1)(b,1)(c,0) \quad (a < b).$$

Such a triangle has area $(b-a)/2$; for $2n$ such triangles to tile a rectangle of area $n$, it is necessary that they all have area $1/2$, i.e. be of the form

$$(a,0)(a+1,0)(c,1) \quad \text{or} \quad (a,1)(a+1,1)(c,0). \tag{1}$$

Let $T_{a,b}$, for $a, b \geq 0$, be the trapezoid (or rectangle, triangle, degenerate segment) having vertices $(0,0)$, $(a,0)$, $(b,1)$, $(0,1)$, and let $t_{a,b}$ be the number of ways to tile it with triangles of type (1). If $a$ and $b$ are positive, the right-hand edge can be covered in only two ways:

- Laying the tile $(a-1,0)(a,0)(b,1)$ and tiling the resulting $T_{a-1,b}$;
- Laying the tile $(a,0)(b-1,1)(b,1)$ and tiling the resulting $T_{a,b-1}$.

Thus in this case $t_{a,b} = t_{a-1,b} + t_{a,b-1}$. This, together with the initial conditions $t_{a,0} = t_{0,b} = 1$, is the recursion of Pascal's triangle; it shows that $t_{a,b} = \binom{a+b}{a}$. Thus the desired answer is $t_{n,n} = \binom{2n}{n}$.

6. Show that

$$a+b+c+\sqrt{3} \geq 8abc\left(\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\right)$$

for all positive real numbers $a$, $b$, $c$ satisfying $ab+bc+ca \leq 1$.

*Solution.* Note that

$$\frac{8abc}{a^2+1} \leq \frac{8abc}{a^2+ab+bc+ca} \leq \frac{8abc}{4a\sqrt{bc}} = 2\sqrt{bc}$$

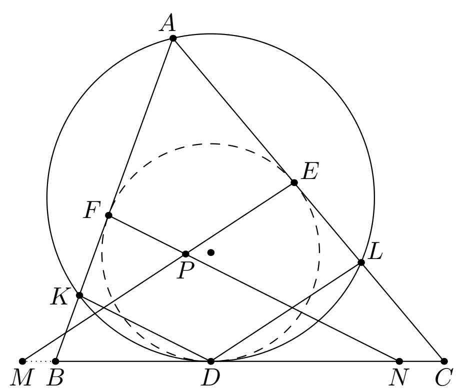
(the second inequality is by AM-GM). Summing up with the analogous inequalities for $b$ and $c$, we find that it suffices to prove that

$$a+b+c+\sqrt{3} \geq 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}).$$

But this easily follows from two applications of Cauchy-Schwarz:

$$a+b+c = \sqrt{a+b+c}\sqrt{b+c+a} \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca}$$
$$\sqrt{3} \geq \sqrt{1+1+1}\sqrt{ab+bc+ca} \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca}.$$

7. The incircle of a triangle $ABC$ touches the sides $BC$, $CA$, and $AB$ at points $D$, $E$, and $F$, respectively. The circle passing through point $A$ and tangent to $BC$ at $D$ intersects the line segments $BF$ and $CE$ at points $K$ and $L$, respectively. The line through $E$ parallel to $DL$ and the line through $F$ parallel to $DK$ intersect at $P$. Let $R_1$, $R_2$, $R_3$, $R_4$ denote the respective circumradii of triangles $AFD$, $AED$, $FPD$, and $EPD$. Prove that $R_1R_4 = R_2R_3$.




*Solution.* Let lines $PE$ and $BC$ intersect at $M$, and let lines $FP$ and $BC$ intersect at $N$. Note that $\angle DAE = \angle CDL = \angle DME$, so $AEDM$ is cyclic and $\triangle EMD$ has circumradius $R_2$. Now by the Extended Law of Sines,

$$\frac{R_4}{R_2} = \frac{PD/(2\sin\angle PED)}{MD/(2\sin\angle MED)} = \frac{PD}{MD}.$$

Likewise, $R_3/R_1 = PD/ND$. So it suffices to prove that $MD = ND$.

By Power of a Point, $BK \cdot BA = BD^2 = BF^2$. So

$$\frac{BA}{BF} = \frac{BF}{BK} = \frac{BN}{BD} = \frac{BN}{BF},$$

implying that $BN = BA$ so $DN = FA$. Likewise $DM = EA = FA$ so we are done.