Berkeley Math Circle
Monthly Contest 2 – Solutions

1. Let p and q be consecutive primes greater than 2. (For instance, p could be 3 and q could be 5; or p could be 103 and q could be 107.) Prove that $p+q$ is the product of at least three (not necessarily different) primes.

   Solution. Since p and q are greater than 2, they are both odd, so $p+q$ is even and immediately factors into 2 and $\frac{p+q}{2}$. Note that $\frac{p+q}{2}$ cannot be prime since it lies between the consecutive primes p and q. Therefore, $\frac{p+q}{2}$ is the product of at least two primes, and $p+q$ is the product of at least three primes.

2. Two players, Cat and Mouse, play the following game on a 4×4 checkerboard. Each player places a checker on a cell of the board (Cat goes first). Then, the two players take turns moving their checkers to an adjacent square, either vertically or horizontally (Cat again goes first). If, after either player's move, the two checkers occupy the same square, Cat wins. Otherwise, if each checker has made 2013 moves without this happening, Mouse wins. Determine, with proof, which player has a winning strategy.

   Solution. Mouse has a winning strategy. Color the cells of the checkerboard black and white alternately, in standard checkerboard fashion, so that adjacent cells are opposite colors. Whichever color Cat places his checker on, Mouse places his checker on a different cell of the *same* color. Then each time Cat moves, the checkers will be on opposite colors, and each time Mouse moves, the checkers will be on the same color. So Cat can never land on Mouse; it remains for Mouse to avoid landing on Cat. Since each cell has at least two neighbors, this is easy to do.

3. Neville Nevermiss and Benjamin Baskethound are two players on the Simpson School basketball team. During Season I, Neville made a higher percentage of his attempted baskets than Ben. The same happened in Season II. Prove or disprove: When the statistics of the two seasons are combined[1], Neville necessarily made a higher percentage of his attempted baskets than Ben.

   Solution. It does *not* follow that Neville made a higher percentage over the two seasons. Consider the following (deliberately extreme) situation. In Season I:

   - Neville attempts 80 baskets and makes 1 of them;
   - Ben attempts 20 baskets and misses them all.

   Then Neville obviously has a higher success rate (1.25% versus 0%). In Season II:

   - Neville attempts 20 baskets and makes them all;
   - Ben attempts 80 baskets and misses 1.

   Then Neville again has the higher success rate (100% versus 98.75%). But overall, both players have attempted 100 baskets, and Ben has made 79% to Neville's 21%!

[1] By combining the statistics, we mean summing the total number of successful baskets and the total number of attempted baskets from both seasons. For example, if Neville scored 35 out of 60 baskets in Season I and 25 out of 40 baskets in Season II, then his aggregate standing is 60 out of 100 baskets. In particular, his percentage of successful baskets is 60%.

4. Find the minimal natural number n with the following property: It is possible to tile the plane with squares whose side lengths belong to the set $\{1, 2, \ldots, n\}$ so that no two squares with the same side length touch along a segment of an edge.

   Remark. Squares with the same side length can touch at a vertex, however. Not all of the side lengths from 1 to n need to be used in the tiling.

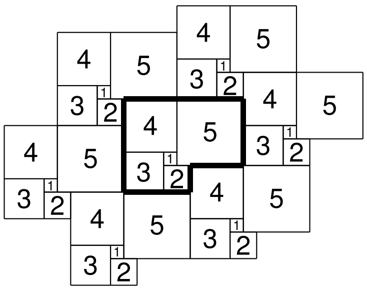


Figure 1

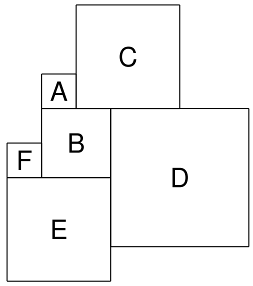


Figure 2

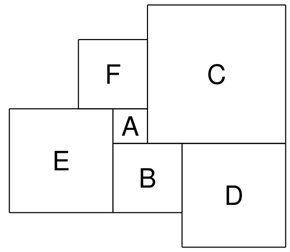


Figure 3

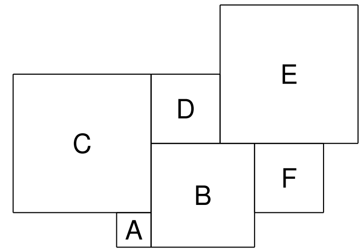
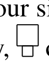
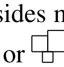
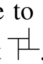


Figure 4

Solution. The answer is $n = 5$. The desired tiling is shown in Figure 1. It is formed by translation from the L-shaped region in bold borders. Since none of the five squares in this region border on squares of like side length, neither does any square in the infinite tiling.

To show that $n \le 4$ does not work, it is necessary to plow through many arrangements of the tiles until reaching a contradiction. We present one method of structuring the argument. Some square in the tiling must have minimal size. Its four sides must be covered by squares larger than itself. If one side is covered by squares that protrude on both ends (pictorially,  or ), then it becomes impossible to cover the opposite side; consequently any minimal square must be covered by four squares in the pinwheel arrangement .

Assume first that the smallest square is the 1×1 and it is surrounded by a 2×2 and a 3×3 in the manner of A, B, C in Figure 2. Now the upper left corner of D must be filled by a 4×4 (a 1×1 would lack the necessary pinwheel layout) and likewise there is a 3×3 at E. Now the corner F cannot be filled with a 4×4 square without wrecking the pinwheel at A, so it must be a 1×1. The space above F is now calling for either a 3×3 or a 4×4, either of which disrupts the pinwheel at A. Thus the arrangement ABC at Figure 2 is impossible.

So if a 2×2 borders on a 1×1, the resulting cavity must be filled by a 4×4 as in ABC of Figure 3. The 3×3 at D is clear. The left side of the A-pinwheel must have a 3×3 (E) since a 4×4 would leave untilable space below B. The remaining spot on the pinwheel is necessarily occupied by a 2×2 (F). Now AFE of Figure 3 is the same configuration as ABC of Figure 2. Hence this case is also impossible. So a 2×2 cannot touch a 1×1.

Thus any 1×1 must be covered alternately by 3×3's and 4×4's as at ABC of Figure 4. The 2×2 at D follows immediately, and since a 1×1 cannot touch a 2×2, we must use a 4×4 at E and a 2×2 at F. Now the space around F must be covered by a 3×3 and a 4×4 which is impossible.

We have left for last the case where there are no 1×1's. Since a tiling using only 3×3's and 4×4's is clearly impossible, the minimal square must be 2×2, covered alternately by 3×3's and 4×4's as at D and CBE of Figure 4 (ignore square A). We derive the 2×2 at F and the resulting contradiction in the same manner as the preceding case.

5. (a) Let a, b, and n be positive integers such that $ab = n^2 + 1$. Prove that

$$|a - b| \geq \sqrt{4n - 3}.$$

(b) Prove that there are infinitely many such triples (a, b, n) such that equality occurs.

Solution. (a) We may assume that $a \geq b$ (thus eliminating the absolute value sign). Write $t = a - b$, so $a = b + t$. Then the equality $n^2 + 1 = ab$ can be rewritten as $n^2 + 1 = (b + t)b$ or $b^2 + tb - (n^2 + 1) = 0$.

Replacing b by x, we consider the quadratic equation $x^2 + tx - (n^2 + 1) = 0$. It has integer coefficients and the integer root $x = b$. Hence the discriminant

$$D = t^2 + 4(n^2 + 1)$$

is a perfect square. Clearly $D > 4n^2 = (2n)^2$. So

$$\begin{aligned} D &\geq (2n + 1)^2 \\ t^2 + 4n^2 + 4 &\geq 4n^2 + 4n + 1 \\ t^2 &\geq 4n - 3 \\ t &\geq \sqrt{4n - 3}. \end{aligned}$$

(b) The construction $a = k^2 + 1$, $b = (k + 1)^2 + 1$, $n = k^2 + k + 1$ (for any integer $k \geq 0$) is easily verified to work.

6. Given $a > b > c > 0$, prove that

$$a^4b + b^4c + c^4a > ab^4 + bc^4 + ca^4.$$

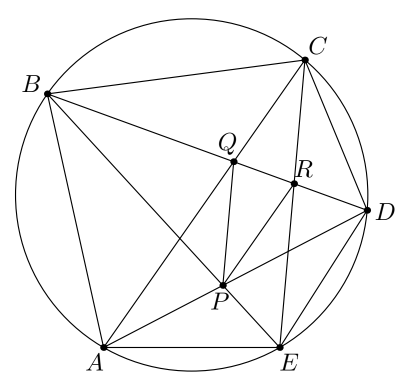
Solution. Observe that equality holds whenever $a = b$, $b = c$, or $a = c$. Knowing this, it is not difficult to factor the difference between the two sides:

$$\begin{aligned} & a^4b + b^4c + c^4a - ab^4 - bc^4 - ca^4 \\ &= (a^4b - ab^4) + (ac^4 - bc^4) + (-a^4c + b^4c) \\ &= (a - b)[(a^3b + a^2b^2 + ab^3) + c^4 + (-a^3c - a^2bc - ab^2c - b^3c)] \\ &= (a - b)[(a^3 - a^3c) + (a^2b^2 - a^2bc) + (ab^3 - ab^2c) + (-b^3c + c^4)] \\ &= (a - b)(b - c)[a^3 + a^2b + ab^2 + (-b^2c - bc^2 - c^3)] \\ &= (a - b)(b - c)[(a^3 - c^3) + (a^2b - bc^2) + (ab^2 - b^2c)] \\ &= (a - b)(b - c)(a - c)[(a^2 + ac + c^2) + (ab + bc) + b^2]. \end{aligned}$$

This last is clearly positive.

Remark. An earlier version of the contest had a typo, where the problem read $a^4b + b^4c + c^4a > ab^4 - bc^4 - ca^4$. This is trivial; notice that $a^4b + b^4c + c^4a > a^4b > ab^4 > ab^4 - bc^4 - ca^4$.

7. Let $ABCDE$ be a cyclic pentagon such that $AB = BC$ and $CD = DE$. Define the intersections $P = AD \cap BE$, $Q = AC \cap BD$, and $R = BD \cap CE$. Prove that $\triangle PQR$ is isosceles.



Solution. Note that

$$\angle APB = \frac{\overset{\frown}{AB} + \overset{\frown}{DE}}{2} = \frac{\overset{\frown}{AB} + \overset{\frown}{CD}}{2} = \angle AQB.$$

Therefore, quadrilateral $ABQP$ is cyclic. Symmetrically, $EDRP$ is cyclic. So

$$\angle PQR = 180 - \angle PQB = \angle BAD = \angle BED = 180 - \angle PRD = \angle PRQ$$

as desired.