Berkeley Math Circle
Monthly Contest 2
Due October 29, 2013

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 2, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Let p and q be consecutive primes greater than 2. (For instance, p could be 3 and q could be 5; or p could be 103 and q could be 107.) Prove that $p+q$ is the product of at least three (not necessarily different) primes.

2. Two players, Cat and Mouse, play the following game on a 4×4 checkerboard. Each player places a checker on a cell of the board (Cat goes first). Then, the two players take turns moving their checkers to an adjacent square, either vertically or horizontally (Cat again goes first). If, after either player's move, the two checkers occupy the same square, Cat wins. Otherwise, if each checker has made 2013 moves without this happening, Mouse wins. Determine, with proof, which player has a winning strategy.

3. Neville Nevermiss and Benjamin Baskethound are two players on the Simpson School basketball team. During Season I, Neville made a higher percentage of his attempted baskets than Ben. The same happened in Season II. Prove or disprove: When the statistics of the two seasons are combined[1], Neville necessarily made a higher percentage of his attempted baskets than Ben.

   Remark. You can try to *prove* this last sentence, by rigorously explaining why it must hold in all situations; or you can try to *disprove* it by describing a situation in which it is false.

4. Find the minimal natural number n with the following property: It is possible to tile the plane with squares whose side lengths belong to the set $\{1, 2, \ldots, n\}$ so that no two squares with the same side length touch along a segment of an edge.

   Remark. Squares with the same side length can touch at a vertex, however. Not all of the side lengths from 1 to n need to be used in the tiling.

5. (a) Let a, b, and n be positive integers such that $ab = n^2 + 1$. Prove that

   $$|a - b| \geq \sqrt{4n - 3}.$$

   (b) Prove that there are infinitely many such triples (a, b, n) such that equality occurs.

[1] By combining the statistics, we mean summing the total number of successful baskets and the total number of attempted baskets from both seasons. For example, if Neville scored 35 out of 60 baskets in Season I and 25 out of 40 baskets in Season II, then his aggregate standing is 60 out of 100 baskets. In particular, his percentage of successful baskets is 60%.

6. Given $a > b > c > 0$, prove that

$$a^4b + b^4c + c^4a > ab^4 - bc^4 - ca^4.$$

7. Let $ABCDE$ be a cyclic pentagon such that $AB = BC$ and $CD = DE$. Define the intersections $P = AD \cap BE$, $Q = AC \cap BD$, and $R = BD \cap CE$. Prove that $\triangle PQR$ is isosceles.