Berkeley Math Circle
Monthly Contest 1
Due October 1, 2013

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 1, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Several ones are written in a row. It is permitted to insert one + or − sign between any two of them or leave the space blank. For example, using six ones one can achieve

$$1\quad 1\quad 1+1-1\quad 1=101.$$

   Is it possible to achieve the result 2013 using

   (a) twenty ones and eight signs;

   (b) twenty ones and nine signs?

   Remark. Each part, (a) and (b), requires a separate answer of *yes* or *no*. If you think the answer is *yes*, you should show how to achieve the result using the appropriate numbers and signs. If you think the answer is *no*, you should rigorously prove that it is impossible.

2. Is it possible to paint each cell of a 4×4 table with one of 8 colors so that for every pair of colors, there are two cells painted with these colors having a common side?

   Remark. Once again, if the answer is *yes*, you should demonstrate the coloring; if the answer is *no*, you should rigorously explain why it is impossible.

3. Ten people sit side by side at a long table, all facing the same direction. Each of them is either a knight (and always tells the truth) or a knave (and always lies). Each of the people announces: "There are more knaves on my left than knights on my right." How many knaves are in the line?

   Remark. Rigorously prove your answer. In particular, you may NOT assume beforehand that the answer is unique.

4. Let ABC be a triangle, and let the bisector of $\angle BAC$ meet BC at D. Let O, O_1, O_2 be the circumcenters of triangles ABC, ABD, and ADC, respectively. Prove that $OO_1 = OO_2$.

5. Determine, with proof, whether there is a function $f(x, y)$ of two positive integers, taking positive integer values, such that

   - For each fixed x, $f(x, y)$ is a polynomial function of y;
   - For each fixed y, $f(x, y)$ is a polynomial function of x;
   - However, $f(x, y)$ does not equal any polynomial function of x and y.

   Remark. A *polynomial function* is an expression built from real numbers and variables using finitely many additions, subtractions, and multiplications; thus

   $$x^4 - 3x + \pi \quad \text{and} \quad \frac{1}{2}(x + y)^2$$

   are polynomials, but

   $$x^y \quad \text{and} \quad \frac{2x}{y^2 + 1}$$

   are not.

6. On the coordinate line, the points with coordinates $1, 2, \ldots, 2n$ are marked, where n is a positive integer. A flea starts jumping from the point with coordinate 1 and after $2n$ jumps returns there, having visited all the marked points. It is known that the total length of all jumps except the last one is $n(2n - 1)$. Find the length of the last jump.

7. Let a, b, c be real numbers between 0 and 1 inclusive. Prove that

   $$\frac{1}{1 + a + b} + \frac{1}{1 + b + c} + \frac{1}{1 + c + a} + a + b + c \leq 3 + \frac{1}{3}(ab + bc + ca)$$