Topology

BMC Notes by Maia Averett

April 26, 2011

A mathematician named Klein
Thought the Möbius band was divine.
Said he, "If you glue
The edges of two,
You'll get a weird bottle like mine.
-Anonymous

1 Surfaces

What shape is the earth? Round? Round like what? Like a pancake? No, it's round like a soccer ball. But how do you know? ...Really, how do you know? Is it because you've seen photos of the earth from space? Well, people figured out that the earth is round long before we figured out how to build rocket ships (or cameras, for that matter!).

As you might have learned in school, scientists as far back as the ancient Greeks theorized that the earth is round. Although they offered no substantive proof of their theories, Pythagoras, Plato, and Aristotle were all supporters of the spherical earth theory, mostly based on the curved horizon one sees at sea. Surely this suggests that the earth is not flat like a pancake, but how can we know that the earth isn't some other round shape, the torus, for example?

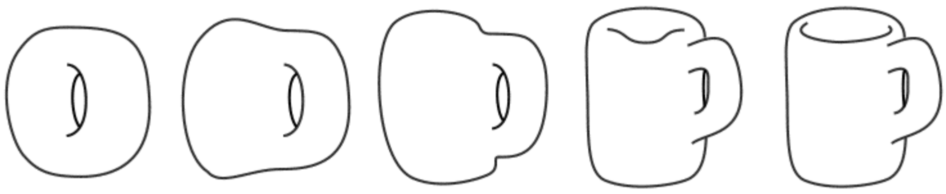
Figure 1: A toroidal Earth. Why not?

Well, if we can walk around the entire earth, then we can come up with plenty of reasons that it's not a torus. The most obvious, perhaps, is that if the earth were a torus, there would be some places where we could stand and look directly up into the sky and see more of the earth! Also, there would be places where the curve of the horizon would be upwards instead of downwards. But how can we really, truly know that the shape is like a ball and not some other strange shape that we haven't yet thought of? An inquisitive person studying this question might begin by assembling all the maps they can find, poring over the overlaps and trying to figure out how to patch them up. Given enough maps to cover the surface of the earth, we can tell that the earth is spherical! We just have to patch together the maps along their overlaps.

This basic idea is exactly the idea that underlies the way mathematicians think about surfaces. Roughly speaking, a *surface* is a space in which every point has a neighborhood that "looks like" a two-dimensional disk (i.e. the interior of a circle, say $\{x, y \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$.)[1] A sphere is an example of a surface, as is the torus. Some of our natural notions of surfaces don't quite fit this definition since they have edges, or places where you could fall off if you weren't careful! Mathematically, these are *surfaces with boundary*: spaces in which every point has either a neighborhood that looks like a two-dimensional disk or half of a two dimensional disk (i.e. $\{x, y \in \mathbb{R}^2 \mid x^2 + y^2 < 1, y \geq 0\}$). The old circular model of the earth where you can sail off the edge is an example of a surface with boundary. Another example is a cylinder without a top and bottom.

Figure 2: A surface with boundary

Now that we know what a surface is, we want to try to figure out what kinds of surfaces there are. From a topologist's perspective, we are interested in the general shape of the surface, not in its size. Although notions of size and distance (geometric notions) are important in reality, topologists seek to understand the coarser shapes as a first approximation. For example, from a topological point of view, a sphere is a sphere, it doesn't matter how large the radius is. We allow ourselves to deform spaces as if they were made of rubber. We'll consider two surfaces to be the same if we can stretch, shrink, twist, push, or wriggle one surface around until it looks like the other surface. But we have to be nice in our deformations: we don't create holes or break or tear any part of our surface. So, an apple would be considered the same as a pear, doesn't matter if it has a big lump on one end. A classic joke along these lines is that a topologist can't tell the difference between a coffee cup and a donut, since if we had a flexible enough donut, we could make a dent in it and enlarge that dent to be the container of the coffee cup, while smushing the rest of the donut down in to the handle of the coffee cup.



Let's begin by trying to make a list of surfaces that we know. What surfaces can you think of? The first one that comes to mind is the surface of the earth: it's a sphere. (Note here that we're only talking about the surface of the earth, not all the dirt, water, oil, and molten rock that make up its insides! Just the surface. Like a balloon.) Another surface that comes up a lot is

[1] A precise definition of a surface can be found in the appendix of this note.

the *torus*, which is shaped like an innertube. For the most part today, we're going to restrict our investigation to compact (which means "small" in the sense that they can be made up of finitely many disks patched together) and connected (made of one piece, i.e. you can walk from one point to every other point on the surface without jumping). We will see some examples of surfaces with boundary because they are surfaces that you may be familiar with. For example, a cylinder without a top or bottom is a surface with boundary. A Möbius strip is a surface with boundary.

2 Representing Surfaces on Paper

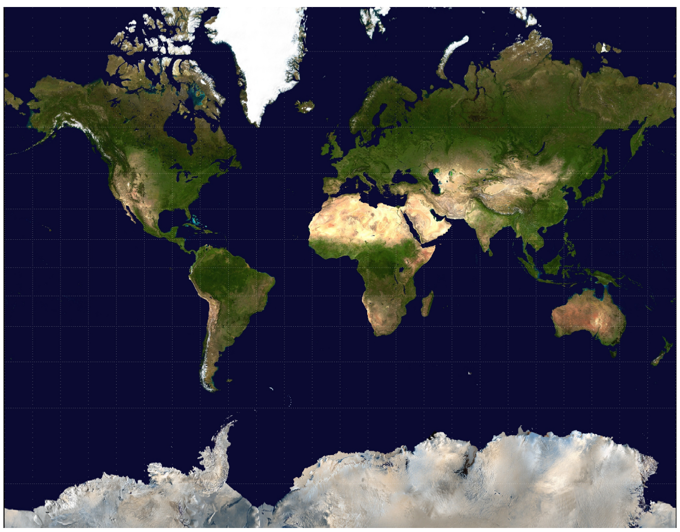
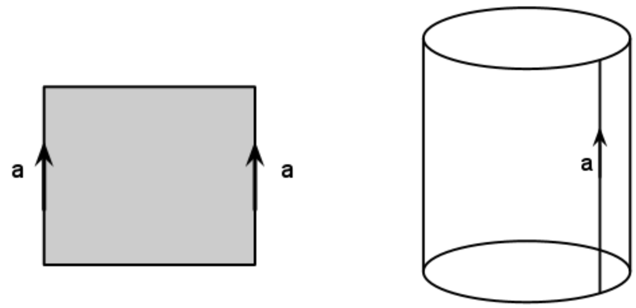


Figure 3: Map of the world

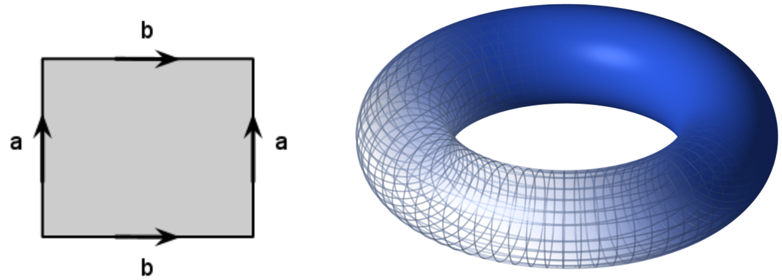
Drawing surfaces on paper or on the blackboard is difficult. However, we'll see that it's easy to record the instructions for making them with a simple diagram on a flat piece of paper.

We take our inspiration from maps of the world. In a typical world map, the globe is split open and stretched a bit so it can be drawn flat. We all understand that if we walk out the right side of the map, we come in through the left side at the same height. This is a pretty useful idea! We can imagine a seam on a globe that represents this edge. We can think of taking the map and gluing up the left and right edge to return to our picture of the globe.

There is one slight dishonesty in the typical world map: the representations of very northernly and southernly parts of the earth aren't very accurate. They're much bigger than they really are! In fact, the entire line at the top edge of the map really represents just a single point on the globe, the north pole. Similarly for the bottom edge and the south pole. We can make a more honest map by shrinking these edges down so that we have one point at the top and one point at the bottom, representing the north and south poles, respectively. Then our resulting picture is a circe! It has the same properties with respect to walking out through the right edge and coming back in through the left. We can record this information by drawing arrows on the boundary of the circle to indicate how we are to glue up the picture to create a globe. It's a very nice and easy to imagine picture: if we glue up one semicircular edge of a circle to the other semicircular edge (without twisting!) then the resulting surface is a sphere. Let's see some more examples of how this works.

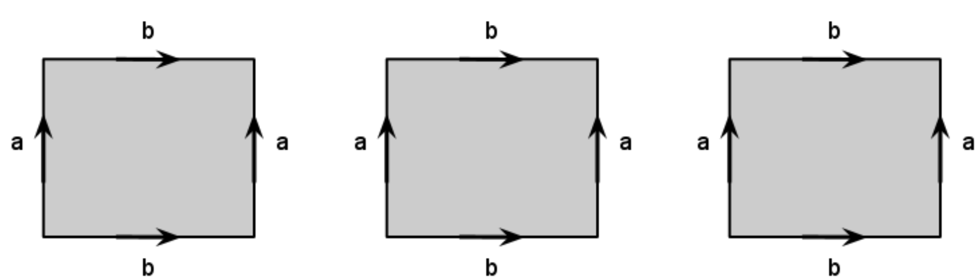


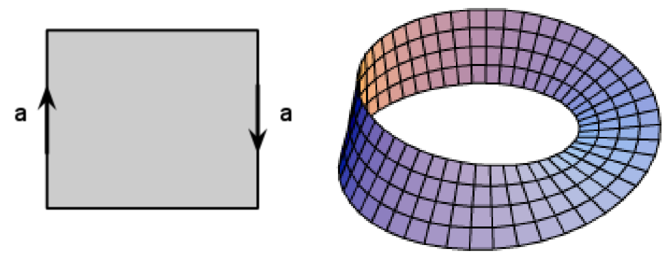
Example 2.1 (The cylinder)**.** We can create a cylinder by using a piece of paper and gluing the ends together. Thus we can write down instructions for making a cylinder by drawing a square and labeling a pair of opposite edges with a little arrow that indicates gluing them together.



Example 2.2 (The torus)**.** Since we are mathematicians, we come up with fancy names for ordinary objects. The surface of an innertube is referred to as the torus. The diagram below represents a gluing diagram for the torus. To see this, first imagine bringing two of the edges together to form a cylinder (without top and bottom, just the sides of a tin can). Since the circle at the top of the can and the circle at the bottom of the can are to be glued together, we can imagine stretching the can around and gluing them to obtain a surface that looks like the surface of a donut. Let's practice thinking about how walking around on the surface is represented on the diagram. If we walk out the left edge, we come back in the right edge at the same height. Similarly, if we walk out the top, we come in the bottom at the same left-right position. It's like PacMan!

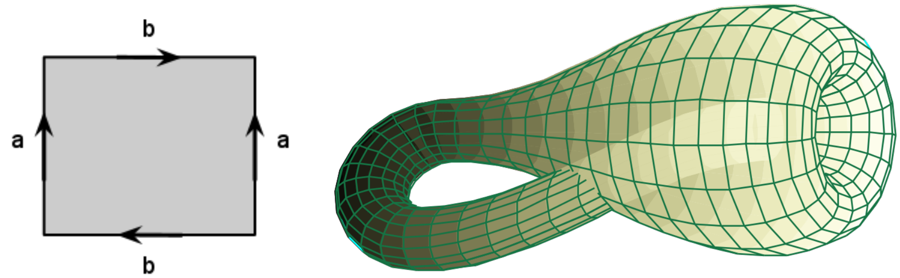
Exercise 2.3. Imagine you are a little two-dimensional bug living inside the square diagram for the torus above. You decide to go for a walk. Trace your path. Be sure to exit some of the sides of the square and be careful about where you come back in! Do this several times. Here are some diagrams to practice on.





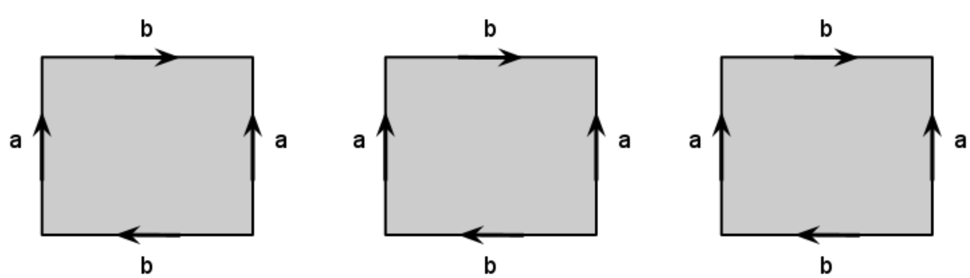
Example 2.4 (The Möbius Strip)**.** What happens if we start with a square and identify a pair of opposite edges, but this time in opposite directions? The resulting surface is a Möbius strip!

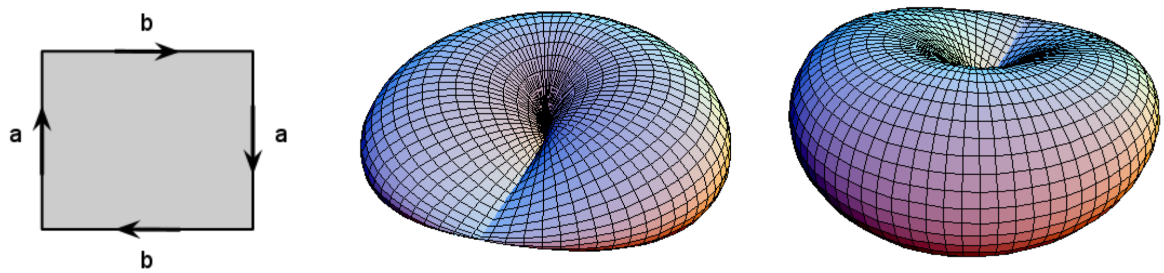
Exercise 2.5. A cylinder has two boundary circles. How many boundary circles does a Möbius strip have?



Example 2.6 (Klein bottle)**.** What happens if we reverse the direction that we glue one of the pairs of edges in the diagram that we had for the torus? We can begin by again gluing up the edges that match up to create a cylinder. But now if we try to stretch it out and glue the boundary circles together, we see that the arrows don't match up like they did for the torus! We can't just glue the circles together because our gluing rule says that the arrows must match up. The only way to imagine this is to imagine pulling one end of the cylinder through the surface of the cylinder and matching up with our circle from the inside. The resulting representation of the surface doesn't look like a surface, but it really is! It's funny appearance is just a consequence of the way we had to realize it in our three-dimensional world.

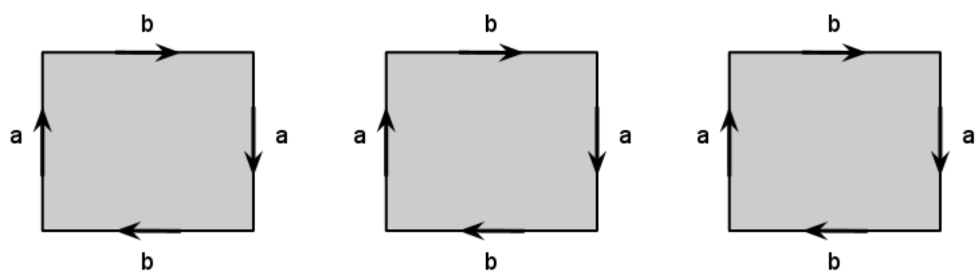
Exercise 2.7. Imagine you are a little two-dimensional bug living inside the square diagram for the Klein bottle above. You decide to go for a walk. Trace your path. Be sure to exit some of the sides of the square and be careful about where you come back in! Do this several times. Here are some diagrams of the Klein bottle to practice on.





Example 2.8 (The projective plane)**.** What happens if we reverse not just one of the pairs, but *both* of the pairs of edges in our diagram for the torus? The resulting surface is called the *projective plane* and it is denoted $\mathbb{R}P^2$. It's hard to imagine what this surface looks like, but our square diagram will allow us to work with it easily!

Exercise 2.9. Imagine you are a little two-dimensional bug living inside the square diagram for $\mathbb{R}P^2$ above. You decide to go for a walk. Trace your path. Be sure to exit some of the sides of the square and be careful about where you come back in! Do this several times. Here are some diagrams to practice on.

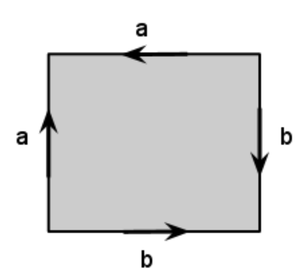


Definition 2.10. A *gluing diagram* for a polygon is an assignment of a letter and an arrow to each edge of the polygon.

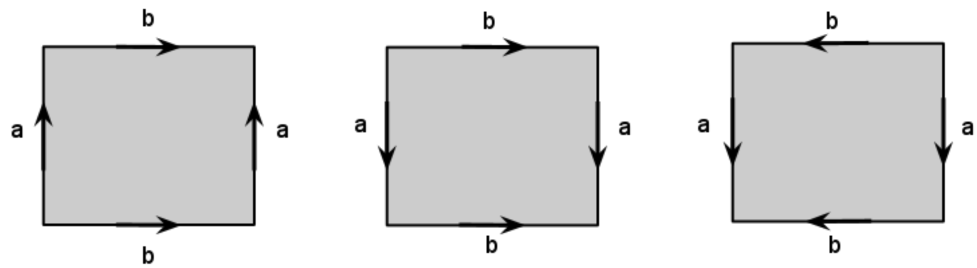
With this general definition, not every gluing diagram represents a surface. For example, if three edges are labeled with the same letter, then these glue up to give something whose cross section looks like $\leftthreetimes$! However, if we assume that the edges are always glued in pairs, then the resulting pasted up object will always be a surface. It's clear that every point in the interior of the polygon has a neighborhood that looks like a disk. A point on one of the edges but not on a corner has a neighborhood that looks like a disk if we think about the corresponding point on the edge that it's glued to and draw half-disks around each of them. A point on one of the corners can similarly be given a neighborhood that looks like a disk.

Example 2.11. The squares that we thought about above for the cylinder, the torus, the Klein bottle, the Möbius strip, and $\mathbb{R}P^2$ are gluing diagrams for these surfaces.

Exercise 2.12. What surface is represented by the gluing diagram below?

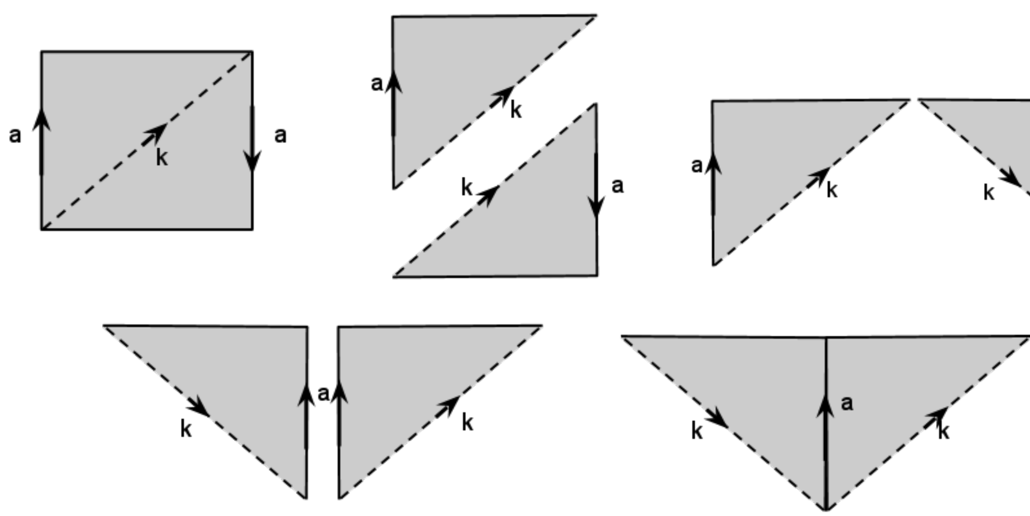


There might be many different diagrams that represent the same surface. For example, we could draw the diagram for the torus in the following ways (and this isn't even remotely all of them!). The important thing for a square to represent the torus is that opposite edges are identified without twists.

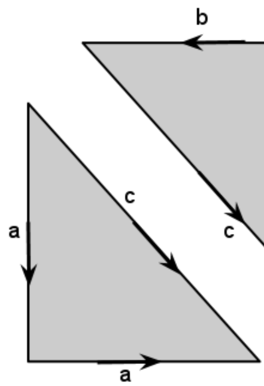


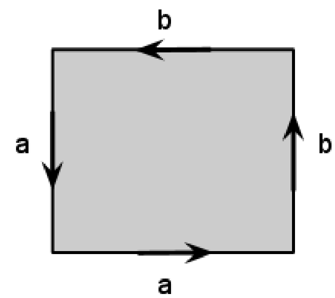
One technique for showing that two gluing diagrams represent the same surface is to take one of the diagrams and cut it and reglue it (possibly repeatedly) until it looks like the other.

Example 2.13 (A Klein bottle is made from two Möbius strips)**.** In this example, we'll show that gluing two Möbius strips together along their boundary circles results in the Klein bottle. This explains the limerick at the beginning of these notes! First, we'll cut and rearrange the gluing diagram for the Möbius strip so that the boundary circle is displayed in one continuous piece.



Now we can see that the top edge of the triangle is the boundary of the Möbius strip, so this makes it easier to take two copies of the Möbius strip (in its new gluing diagram) and glue them together along their boundary circles (the boundary circles are labeled c in the diagram below on the left).

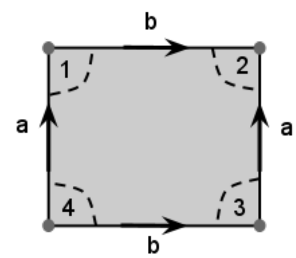




Hrmm.. This doesn't quite look like our standard diagram for the Klein bottle! Your job in the next problem is to figure out how to cut it and rearrange the pieces so that it looks like the standard diagram.

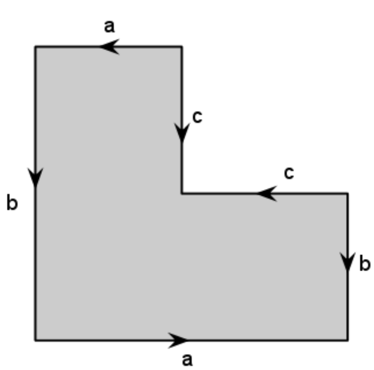
Problem 2.14. Use cutting and regluing techniques to show that the gluing square above right represents the Klein bottle. *Hint:* Cut along a diagonal.

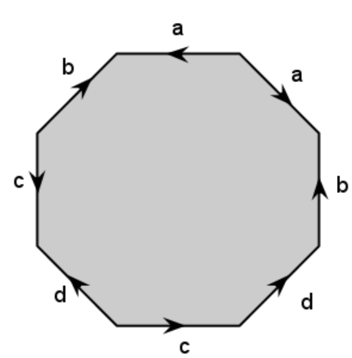
Problem 2.15. In the standard gluing diagram for the torus, all four corners represent the same point in on the surface of the torus. Cutting out a disk around this point is the same as cutting out the corners in the gluing diagram. Paste together the corners $1, 2, 3$, and 4 so they form a disk. Do the same for a Klein bottle. What happens for $\mathbb{R}P^2$?

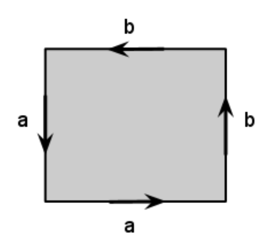


Problem 2.16. What surface results from gluing a disk to the boundary circle of a Möbius band?

Problem 2.17. Which of the following diagrams represent equivalent surfaces? (Note that each diagram represents its own surface. It is not intended that you glue all the a's together, etc, but only the ones on that specific diagram.)



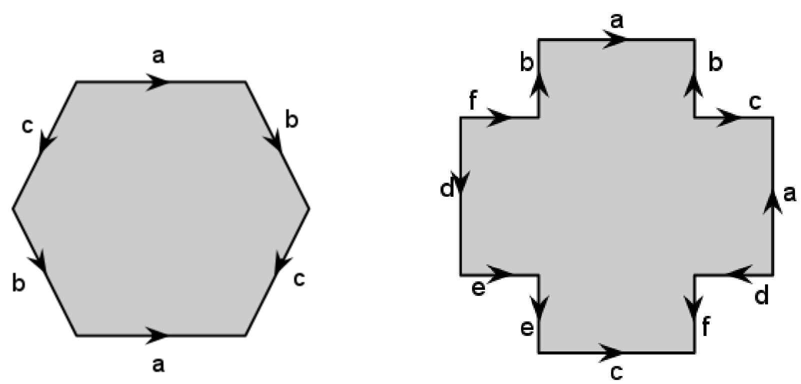




In a gluing diagram, we identify the edges of a polygon. This means that sometimes, the corners of our polygon are not distinct points. For example, in the standard square diagram for the torus, all four corners really represent the *same* point in the surface.

Exercise 2.18. Which corners in the standard square diagram for the Klein bottle represent distinct points in the surface? What about in the standard square for $\mathbb{R}P^2$?

Exercise 2.19. In each of the following diagrams, identify which corners represent the same point and which are distinct.



Problem 2.20. Since we are topologists, we don't care so much whether lines are straight or curved. We could also think about gluing diagrams that result from dividing a circle into subsegments (edges) and assigning letters and arrows to these edges. Our example of the circular world map is a gluing diagram for the sphere S^2 as a circle divided into two edges. Find a similar diagram for $\mathbb{R}P^2$.

One way to record the gluing is by writing down a word that describes what letters we see when we walk around the edges of the gluing diagram. Begin at one corner of the diagram and walk around the perimeter of the diagram. When we walk along an edge labeled with a letter, say a, in the same direction as its assigned arrow, we write that letter. If we walk along an edge labeled with a letter, say a, but in the opposite direction of its assigned arrow, we write down a'. The string of letters contains the same information as the gluing diagram, so long as we remember the code that translates between the words and the gluing diagram.

Exercise 2.21. Draw the gluing diagrams associated with the following words: $abab$, $abca'b'c'$, $aba'b$, $ba'ba'$, $ab'ab$, $bacc'b'a$.

Problem 2.22. Do any of the words in the previous exercise represent the same surface?

Problem 2.23. Consider gluing diagrams for a square that glue together pairs of edges. Let's use the letters a and b to denote the pairs of edges.

(a) How many are there? *Hint:* To count them, you need to keep track of the letter of each edge and also its direction. Use the idea above of walking around the edge and recording the word you walk along. So, this is really a question that asks: how many four letter words are that use the letters a, a', b, b' such that both a and b appear exactly twice (where twice means with or without the decoration $'$, e.g. you could have a and a, or a and a', or a' and a' in your list, but you cannot have a appearing only once or three times).

(b) But this number is clearly too big if we want to count the different surfaces represented by these gluing diagrams. For example, if one diagram can be obtained from another by rotating it a quarter turn to the right, then these must represent the same surface. Similarly, if one diagram can be obtained from another by flipping the square over, they also must represent the same surface. By rotating and flipping our diagrams, we can reduce to the case where the left edge of the square is labelled with a and the arrow points up. Convince yourself that any gluing diagram for the square can be flipped and rotated so that it is in this position.

(c) Now that we've determined that we can reduce to the case where the left edge of the square is labelled with a and with an upward pointing arrow, try to make a complete list of gluing diagrams that doesn't have any "obvious" repeats. By "obvious," I mean there isn't a sequence of rotations and a flip that will take one diagram on your list to another.

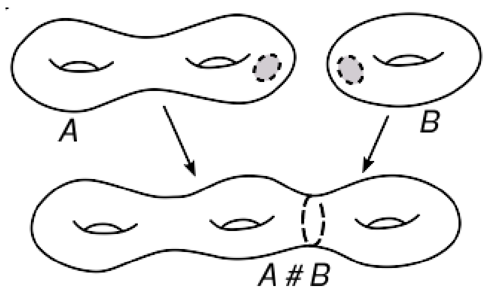
(d) Can you identify any of the diagrams as surfaces that we know?

(e) Peek to the last page of these notes to see the complete list and to check your answers. Two of our diagrams turn out to represent the Klein bottle and two represent the projective plane $\mathbb{R}P^2$. Find a way to cut and paste the non-standard diagrams of the Klein bottle and $\mathbb{R}P^2$ so that they look like the standard ones.

3 Connected Sum

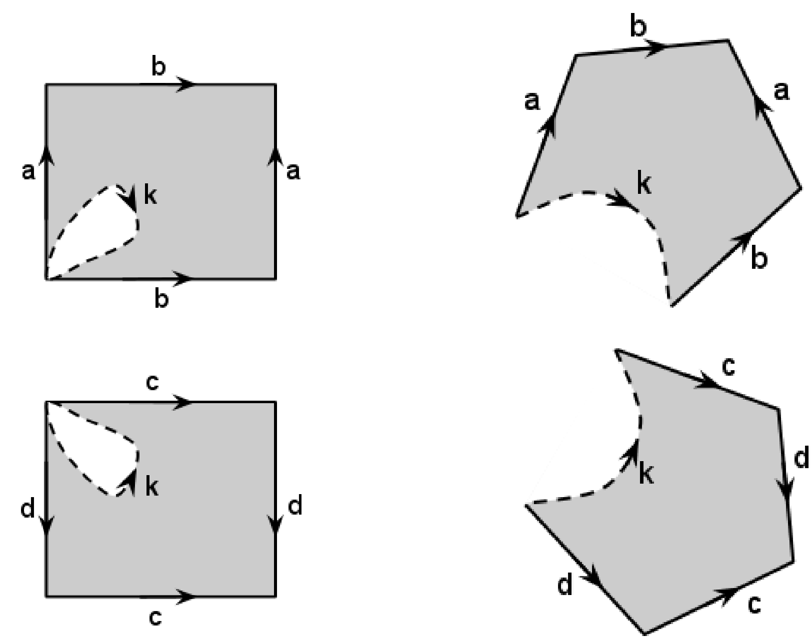
Thinking about a donut with two holes, you might immediately imagine it as two donuts stuck together. This is the idea behind the connected sum of surfaces. The connected sum will give us a way of building more complicated surfaces out of the simpler ones that we already have.

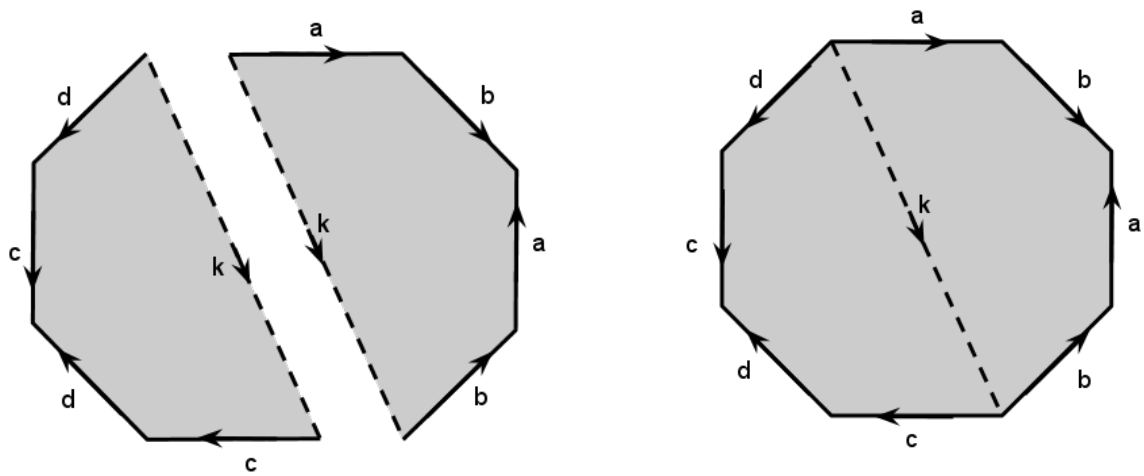
Definition 3.1 (Connected Sum)**.** Given surfaces A and B, the *connected sum* of A and B, denoted $A\#B$, is formed by cutting a disk from A and a disk from B and gluing the surfaces together along the boundary.



The connected sum is relatively easy to visualize for tori, but for our surfaces that aren't so easy to draw, we would love to have a way of denoting the connected sum in our gluing diagrams. Here's how. Beginning at a corner in the diagram for A, cut out a little loop and orient it with an arrow. Do the same at a corner of B. Now open up the surface at that corner so that the loop becomes a straight line. Glue the surfaces together along the line. Now you can reshape and redraw your gluing diagram so that it is more regular and prettier, if you like.

Example 3.2. The pictures below show how this works for two tori $T\#T$. I've chosen the labeling on the torus diagrams to make the resulting connect sum pretty, but you could begin with any diagram for the torus.





Problem 3.3. Following the steps outlined above, find a polygon that represents $T\#T\#T$. Now do $T\#T\#T\#T$. Generalize this to the connected sum of n tori.

Problem 3.4. Show that $\mathbb{R}P^2\#\mathbb{R}P^2 = K$.

Problem 3.5. In this problem, we'll show that $T\#\mathbb{R}P^2 = K\#\mathbb{R}P^2$.

(a) Express $T\#\mathbb{R}P^2$ and $K\#\mathbb{R}P^2$ as gluing diagrams for hexagons. You'll need the "two-edged" gluing diagram for $\mathbb{R}P^2$ from Problem 2.20.

(b) Cut the hexagon for $T\#\mathbb{R}P^2$ along a diagonal so that you can glue together the edges that represented $\mathbb{R}P^2$.

(c) Take your answer from above and cut off a triangluar disk and paste it back along another edge.

Problem 3.6. Use gluing diagrams to prove that connected sum with the sphere S^2 returns the same surface. *Hint:* First draw a circular gluing diagram for S^2.

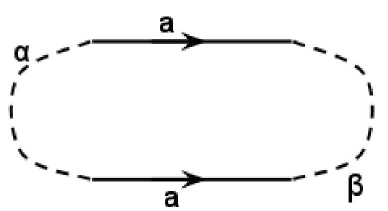
In the following sequence of problems, we work towards the classification of surfaces by studying the gluing diagrams in a systematic way.

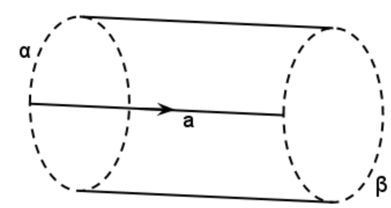
Problem 3.7. Suppose a gluing diagram for a surface has two consecutive edges labeled with the same letter in opposing directions. Show that this pair of edges represents connect sum with S^2 and can thus be eliminated.

Problem 3.8. Suppose a gluing diagram for a surface has two consecutive edges labeled with the same letter in the same direction. Show that this pair of edges represents a connect sum with $\mathbb{R}P^2$.

Problem 3.9. Suppose a gluing diagram for a surface has two nonconsecutive edges labeled with the same letter in the same direction. Find a way to cut and reglue the diagram so that the two edges are consecutive, thus showing that this also represents a connect sum with $\mathbb{R}P^2$.

Problem 3.10. Suppose a gluing diagram for a surface S has two nonconsecutive edges labeled with the same letter a in opposing directions. These edges divide the boundary of the gluing diagram into two distinct pieces. Let's call one piece α and one piece β. In the pictures below, I've used a dotted curve to represent these pieces because we don't yet want to make any assumptions on what they look like.



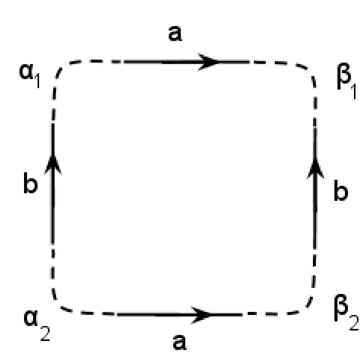


(a) Gluing these edges together results in a cylindrical object whose boundary consists of the remaining edges of the gluing diagram, so one boundary piece of the cylinder is α and the other is β. Suppose all edges in α are glued to other edges in α and similarly for β, so that no edge from α is glued to any edge in β. Find a way to cut the cylinder and then glue in disks to create two gluing diagrams for surfaces: one whose boundary is α and one whose boundary is β. Notice that you've decomposed S into two surfaces S_α and S_β such that $S_\alpha \# S_\beta = S$. Moreover, both S_α and S_β have fewer edges than S. (This will be important for an induction argument later.)

(b) Now suppose some edge in α is glued to some edge in β. Let's call that edge b.

(i) If both occurrences of b are oriented in the same direction, then this represents a connect sum with $\mathbb{R}P^2$ by Problem 3.9. Thus we can split off this $\mathbb{R}P^2$ and the resulting two gluing diagrams have fewer edges. (There's nothing to prove in this part of the problem. Just convince yourself that these statements are true.)

(ii) If the occurrences of b are oriented in opposite directions, we'll show that this represents connect sum with a torus. We can draw our diagram as in the picture to the right. Here α_1 and α_2 are the pieces of α not containing b. Similarly for β_1 and β_2.



- Draw a picture of the cylinder that results from gluing along a.
- Now bend the cylinder around so that you can glue along b. This will result in a torus minus a disk, where the boundary is comprised of $\alpha_1, \alpha_2, \beta_1$, and β_2.

- Now draw a circle around the missing disk, leaving a bit of space between the edges of the disk and your circle. Cut this part out of the surface. And draw it separately on your paper (it's an annulus with outside boundary your cut and inside boundary $\alpha_1, \alpha_2, \beta_1$, and β_2.
- Flip the disk inside out so that the cut is on the inside and $\alpha_1, \alpha_2, \beta_1$, and β_2 are on the outside. Explain why this construction has split off a torus via undoing connect sum.

Problem 3.11. Putting together the previous problems, we have almost shown that every gluing diagram of a $(2n)$-gon whose edges are identified in pairs can be reduced to a connected sum of torii or a connected sum of $\mathbb{R}P^2$'s. Formalize the argument using induction, and carefully citing the relevant problems.

4 Formal Definitions

In this section, we record the formal definitions, for the interested reader.

Definition 4.1 (Topological space)**.** A *topological space* is a set X together with a collection of subsets τ satisfying the following conditions:

(i) The empty set and X are in τ.

(ii) The union of any collection of sets in τ is again in τ.

(iii) The intersection of any finite collection of sets in τ is again in τ.

The collection τ is called the *topology* on X. The elements of X are generally referred to as *points* and the elements of τ are referred to as *open sets*.

Definition 4.2 (Continuous map)**.** If (X, τ) and (Y, σ) are topological spaces, a function $f : X \to Y$ is said to be continuous if $f^{-1}(U) \in \tau$ for all $U \in \sigma$.

Definition 4.3 (Hausdorff)**.** A topological space (X, τ) is *Hausdorff* if for every $x_1, x_2 \in X$, there exist $U_1, U_2 \in \tau$ such that $x_i \in U_i$ and $U_1 \cap U_2 = \emptyset$.

Definition 4.4 (Basis)**.** A basis for a toplogy τ on a set X is a collection $B = \{U_\lambda \in \tau \mid \lambda \in \Lambda\}$ such that every other set $U \in \tau$ can be written as a union of elements of B.

Definition 4.5 (Second-countable)**.** A topological space is *second-countable* if it has a countable basis.

Definition 4.6 (Homeomorphism)**.** A continuous bijection $f : X \to Y$ between topological spaces X and Y is called a *homeomorphism*. If a homeomorphism exists between X and Y, we say that X and Y are *homeomorphic*.

Definition 4.7 (Manifold)**.** A *n-dimensional manifold* is a second-countable Hausdorff topological space together with a collection of open sets $\{U_\alpha\}$, such that $M = \bigcup_\alpha U_\alpha$, and homeomorphisms $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$. The maps φ_α are called *charts*.

Definition 4.8 (Manifold with boundary)**.** A *n-dimensional manifold with boundary* is a second-countable Hausdorff topological space together with a collection of open sets $\{U_\alpha\}$, such that $M = \bigcup_\alpha U_\alpha$, and homeomorphisms $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ or $\varphi_\alpha : U_\alpha \to \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (x_1, \ldots, x_n) \text{ and } x_1 \geq 0\}$. The maps φ_α are called *charts*.

All possibilities for gluing the square are shown below. They are the Klein bottle, $\mathbb{R}P^2$, the sphere, the Klein bottle, the torus, and $\mathbb{R}P^2$, respectively.