Probability and games

Berkeley Math Circle
October 23, 2012

Today we are going to introduce you to the basic concepts of probability. After going through some basic concepts and assumptions, we will spend the first hour solving problems, typically about computing the probability of certain events. In the second hour we will talk about some well-known classical problems and play some games related to them.

Summary / Basics

- Events: if A and B are events, then
  - A^c: the complement of A;
  - $A \cup B$: the union of events A and B; read: "A or B";
  - $A \cap B$: the intersection of events A and B; read: "A and B";
  - A and B are said to be *mutually exclusive* if $A \cap B = \emptyset$.
- Probabilities:
  - $\mathbb{P}(A)$: probability of event A; $0 \leq \mathbb{P}(A) \leq 1$.
  - $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.
  - $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$;
    if A and B are *mutually exclusive*, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.
  - If A and B are *independent*, then $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$.
  - Conditional probability: $\mathbb{P}(A \mid B)$, read: "the probability of A given B";

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Introductory problems

1. Flipping coins. Suppose you flip a fair coin 5 times in a row.

- What is the probability that you see $HTTHT$? How about $TTTTT$?
- What is the probability that you see exactly 2 heads? What is the probability that you see at least one tail? What is the probability of seeing at least 2 heads and at least 2 tails?
- What is the probability of seeing 3 heads in a row somewhere in the sequence?

2. Rolling dice. Suppose you roll two indistinguishable dice.

- What is the probability that you see two 3's? What is the probability that you see a 1 and a 6?
- What is the probability that the sum of the two numbers you see is 7? What is the probability that the sum is at least 11? What is the probability that the sum is even?
- What is the probability that one of the numbers is at least three times the other?

Now suppose you roll a red and a blue die.

- What is the probability that you see two 3's? What is the probability that the blue die shows a 1 and the red die shows a 6?
- What is the probability that the sum of the two numbers you see is 7? What is the probability that the sum is at least 11? What is the probability that the sum is even?
- What is the probability that the number on the blue die is at least three times the number on the red die?

3. Drawing balls from urns. Suppose I have an urn with 3 red balls, 5 blue balls, and 7 yellow balls. I put my hand in and draw a ball at random. Without putting the ball back I draw another ball, and then I do this once again. So I have drawn three balls without replacement.

- What is the probability that all three balls are red? All are blue? All are of the same color?
- What is the probability that at least one of the three balls is blue?
- What is the probability that all three balls have different color?
- How do these probabilities change if we always replace the balls after drawing them?

4. Poker. Compute the probability of each poker hand. That is, you are given 5 different cards at random from a standard set of 52 cards ($\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit : 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\}$), which have been well-shuffled. What is the probability that you have a pair? Two pairs? Three of a kind? A straight? A flush (all five cards of the same suit)? A full house (three of a kind and a pair)? Four of a kind? Straight flush? Royal flush ($10, J, Q, K, A$ of the same suit)?

5. Passe-dix. Throwing a die three times, what is the probability that the sum of the three numbers is greater than 10? *Note:* this was the condition for winning in the game of chance *passe-dix*, which was popular in the 17^{th} century.

6. de Méré's problem. What is more probable: (a) throwing at least one "1" in four rolls of a single 6-sided die; (b) throwing at least one "double-one" in 24 throws of two dice? *Note:* this is the problem of 17^{th} century French nobleman and gambler Chevalier de Méré, who approached Blaise Pascal in 1654 to find out the answer.

7. Newton-Pepys problem. What is more probable: (a) throwing at least one "6" in six rolls of a single 6-sided die; (b) throwing at least two "6" in 12 throws of a single 6-sided die? *Note:* this problem arose in the correspondence between Isaac Newton and Samuel Pepys in 1693. Pepys wasn't convinced by Newton's (correct) argument.

8. Rooks on a chessboard. Suppose we place 8 rooks randomly on a chessboard. What is the probability that none of them can capture each other, that is, no column or row contains more than one rook?

9. Lottery. In the Hungarian lottery, 5 numbers are chosen randomly from $\{1, 2, \dots, 90\}$. What is the probability of winning the jackpot (getting all 5 numbers correct)? What is the probability of not getting any numbers correct? What is the probability of having 1, 2, 3, or 4 hits?

10. Finding the correct key. Suppose you have a keyring with n keys, and you want to open a door. Exactly one of the keys opens the door, but you don't know which one. You try the keys one by one, in a random order without repetition, until you find the correct key. What is the probability that you open the door on the first try? On the second try? On the last try?

11. A game with a die and two coins. Alice and Bob play the following game. Alice rolls the die. Then she flips a pair of coins the number of times that the die shows. If there is an "HH" (double heads) among the coin flips, then Alice wins and gets a dollar from Bob, otherwise Bob wins and gets a dollar from Alice. For whom is this game beneficial?

Birthday problem. What is the probability that among n randomly chosen people, some pair will have the same birthday? For what n is this at least 0.5? For what n is this 0.99? For what n is it sure that there will be two people with the same birthday?

Monty Hall problem. Suppose you're on a game show and you are given the choice of three doors: behind one of them is a car, and behind the other two there are goats. You pick a door. Then the host, who knows what is behind each door, opens another door that has a goat in it. There are then two remaining doors. The host offers you a chance to switch to the other remaining door. Is it to your advantage to switch?

Secretary problem (phrased slightly differently). Princess Jasmine has the chance to choose among n male concubines according to the following rules. The men are led in front of Princess Jasmine one-by-one, in a random order. After meeting each man, Jasmine has to decide whether to keep him or not; she cannot go back to previously seen men, and she does not know anything about the men she hasn't seen yet. (Her palace only has room for one more concubine.) We assume that the men can be ordered unambiguously according to their beauty. Jasmine wants to maximize her probability of choosing the most beautiful man. What is her optimal strategy?