1 Warmup Problems

We probably won't have time to do all of these, but I'd be very interested to hear your ideas about any of them.

Problem 1 At a movie theater, the manager announces that a free ticket will be given to the first person in line whose birthday is the same as someone in line who has already bought a ticket. You have the option of getting in line at any time. Assuming that you don't know anyone else's birthday, and that birthdays are uniformly distributed throughout a 365-day year, what position in line gives you the best chance of being the first duplicate birthday?

Problem 2 There are 29 trees in a long row. Your pet squirrel is hiding in one of them, out of sight in the top branches. Once each minute, you pick one of the trees (any one of them) and climb it. If he's there, you find him.

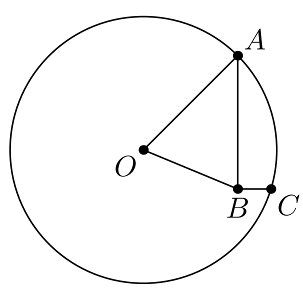
If he's NOT there, you climb down. As you do, your squirrel leaps from the trees he is in to an adjacent tree (He has two choices unless he's at one of the trees at the end of the row, in which case he must move to the only available tree) and hides again. In the next minute, you again may pick any one of the trees and climb it, repeating the process. After each unsuccessful search, your squirrel moves to an adjacent tree. (He never stays put!)

If you keep trying, and use a good strategy, can you be certain to eventually find the squirrel? Can you find him in 29 tries? in 30 tries? in 100 tries? or can he keep evading you forever no matter what?

Problem 3 You are blindfolded, and handed a deck of (52) cards, of which 29 cards are face up and 23 cards are face down. You don't know which cards are face up and which are face down. Your task is to divide the deck into two stacks, in which both stacks have the same number of face-down cards. (You may count cards into stacks, and flip individual cards over, transposing them from face up to face down, and vice versa, but you don't know which cards are face up and which cards are face down.)

Problem 4 (2004 Missouri Collegiate Mathematics Competition) A chess position possesses the following property: On every vertical column and on every horizontal row, there is an odd number of pieces. Prove that there is an even number of pieces on black squares.

Problem 5 (1983 AIME) A machine shop cutting tool is in the shape of a notched circle, as shown. The radius of the circle is $\sqrt{50}$ cm, the length of AB is 6 cm, and that of BC is 2 cm. The angle ABC is a right angle. Find the square of the distance (in centimeters) from B to the center of the circle.



Problem 6 1. (Putnam 2004) Basketball star Shanille OKeals team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her firrst N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

2 AMC Intro

- *Brief* History of the AMC 10 from 1950 through 2012
- what happens next (120 or 2.5% qualify for AIME), 240 or so will take USAJMO.
- Content of the test

  > The AMC 10 covers mathematics normally associated with grades 9 and 10. To challenge students at all grade levels, and with varying mathematical skills, the problems range from fairly easy to extremely difficult. Approximately 12 questions are common to the AMC 10 and AMC 12. The AMC 10 assumes knowledge of elementary algebra; basic geometry knowledge including the Pythagorean Theorem, area and volume formulas; elementary number theory; and elementary probability. What are excluded are trigonometry, advanced algebra, and advanced geometry.

  Statistics on AMC FAQ comparing AMC 10 questions to NCTM Standards suggest: 40% "Algebra", 40% "Geometry", and the rest a smattering of "Number Operations", "Data and Probability", and "Problem Solving". But what does that mean?

  For our purposes, the main content can be broken down into four categories: Algebra, Number Theory, Combinatorics, and Geometry. We'll try to itemize the most common mathematical ideas that show up repeatedly on the test as we go through representative problems.

- AMC 10 vs. AMC 12?
- Statistics
- How to take the AMC 10
  - read carefully! Be sure you understand what the question is asking for.
  - If you know the right theorem or concept, great! If not, you may still be able to do the problem if you are careful and systematic [especially in counting problems]
  - If you can safely add simplifying assumptions to a problem, give it a try! (If something should work for any quadrilateral, try a square). Beware, though, be sure your simplifying assumption is justified.
  - If you're stuck, use the fact that the problems are multiple choice.
- (Taking the contest at Stanford – probably not relevant here)

21. (2003 AMC 10 B, ♯24) The first four terms in an arithmetic sequence are $x+y$, $x-y$, xy, and x/y, in that order. What is the fifth term?

(A) $-\frac{15}{8}$ (B) $-\frac{6}{5}$ (C) 0 (D) $\frac{27}{20}$ (E) $\frac{123}{40}$

22. *(1974 AHSME, ♯21)* In a geometric series of positive terms the difference between the fifth and fourth terms is 576, and the difference between the second and first terms is 9. What is the sum of the first five terms of this series?

(A) 1061 (B) 1023 (C) 1024 (D) 768 (E) none of these

23. *(1981 AHSME, ♯14)* In a geometric sequence of real numbers, the sum of the first two terms is 7, and the sum of the first six terms is 91. The sum of the first four terms is

(A) 28 (B) 32 (C) 35 (D) 49 (E) 84

24. *(1994 AHSME, ♯20)* Suppose x, y, z is a geometric sequence with common ratio r and $x \neq y$. If x, $2y$, $3z$ is an arithmetic sequence, then r is:

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 2 (E) 4

25. *(2002 AMC 10 B, ♯23)* Let $\{a_k\}$ be a sequence of integers such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$ for all positive integers m and n. Then a_{12} is:

(A) 45 (B) 56 (C) 67 (D) 78 (E) 89

26. *(2004 AMC 10 A, ♯24)* Let $a_1, a_2, \ldots$ be a sequence with the following properties:

(i) $a_1 = 1$, and
(ii) $a_{2n} = n \cdot a_n$ for any positive integer n.

What is the value of $a_{2^{100}}$?

(A) 1 (B) 2^{99} (C) 2^{100} (D) 2^{4950} (E) 2^{9999}

27. *(2006 AMC 10 B, ♯18)* Let $a_1, a_2, \ldots$ be a sequence for which

$$a_1 = 2, \ \ a_2 = 3, \ \text{ and } \ a_n = \frac{a_{n-1}}{a_{n-2}} \text{ for each positive integer } n \geq 3.$$

What is a_{2006}?

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) 2 (E) 3

28. *(2000 AMC 10A, ♯24)* Let f be a function for which $f(x/3) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

(A) $-\frac{1}{3}$ (B) $-\frac{1}{9}$ (C) 0 (D) $\frac{5}{9}$ (E) $\frac{5}{3}$

29. *(2002 AMC 10 A, ♯14)* Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. The number of possible values of k is:

(A) 0 (B) 1 (C) 2 (D) 4 (E) more than 4

30. *(2003 AMC 10A, ♯18)* What is the sum of the reciprocals of the roots of the equation:

$$\frac{2003}{2004}x + 1 + \frac{1}{x} = 0\,?$$

(A) $-\frac{2004}{2003}$ (B) -1 (C) $\frac{2003}{2004}$ (D) 1 (E) $\frac{2004}{2003}$

31. *(1975 AHSME, ♯22)* If p and q are primes and $x^2 - px + q = 0$ has distinct positive integral roots, then which of the following statements are true?

(I) The difference of the roots is odd.
(II) At least one root is prime.
(III) $p^2 - q$ is prime.
(IV) $p + q$ is prime.

(A) I only (B) II only (C) II and III only (D) I, II, and IV only

(E) All are true

32. *(2010 AMC 10 A, ♯21)* The polynomial $x^3 - ax^2 + bx - 2010$ has three positive integer zeros. What is the smallest possible value of a?

(A) 78 (B) 88 (C) 98 (D) 108 (E) 118

33. *(2002 AMC 10 A, ♯16)* If $a+1 = b+2 = c+3 = d+4 = a+b+c+d+5$, then $a+b+c+d$ is:

(A) -5 (B) $-10/3$ (C) $-7/3$ (D) $5/3$ (E) 5

34. *(2002 AMC 10 B, ♯20)* Let a, b, and c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. Then $a^2 - b^2 + c^2$ is:

(A) 0 (B) 1 (C) 4 (D) 7 (E) 8

6. *(2002 AMC 10 B, ♯6)* For how many positive integers n is $n^2 - 3n + 2$ a prime number?

   (A) none (B) one (C) two (D) more than two, but finitely many

   (E) infinitely many

7. *(2002 AMC 10 A, ♯15)* The digits 1, 2, 3, 4, 5, 6, 7, and 9 are used to form four two-digit prime numbers, with each digit used exactly once. What is the sum of these four primes?

   (A) 150 (B) 160 (C) 170 (D) 180 (E) 190

8. *(2000 AMC 10, ♯11)* Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could not be obtained?

   (A) 21 (B) 60 (C) 119 (D) 180 (E) 231

9. *(2002 AMC 10 B, ♯15)* The positive integers A, B, $A - B$, and $A + B$ are all prime numbers. The sum of these four primes is:

   (A) even (B) divisible by 3 (C) divisible by 5 (D) divisible by 7 (E) prime

10. *(2010 University of South Carolina High School Mathematics Contest, ♯14)* How many integers between 1 and 1000 have exactly 27 positive divisors?

    (A) 0 (B) 1 (C) 2 (D) 27 (E) 28

11. *(2002 AMC 10 B, ♯16)* For how many integers n is $\dfrac{n}{20-n}$ the square of an integer?

    (A) 1 (B) 2 (C) 3 (D) 4 (E) 10

12. *(2007 AMC 10 B, ♯25)* How many pairs of positive integers (a, b) are there such that a and b have no common factors greater than 1 and

$$\frac{a}{b} + \frac{14b}{9a}$$

    is an integer?

    (A) 4 (B) 6 (C) 9 (D) 12 (E) infinitely many

13. *(2002 AMC 12 B, ♯13)* The sum of 18 consecutive positive integers is a perfect square. The smallest possible value for this sum is:

    (A) 169 (B) 225 (C) 289 (D) 361 (E) 441

14. *(2005 AMC 10 A, ♯21)* For how many positive integers n does $1 + 2 + \cdots + n$ evenly divide $6n$?

    (A) 3 (B) 5 (C) 7 (D) 9 (E) 11

15. *(2005 AMC 10 B, ♯22)* For how many positive integers n less than or equal to 24 is $n!$ evenly divisible by $1 + 2 + \cdots + n$?

    (A) 8 (B) 12 (C) 16 (D) 17 (E) 21

16. *(2005 AMC 10 A, ♯15)* How many positive cubes divide $3! \cdot 5! \cdot 7!$?

    (A0 2 (B) 3 (C) 4 (D) 5 (E) 6

17. *(2010 AMC 12 B, ♯9)* Let n be the smallest positive integer such that n is divisible by 20, n^2 is a perfect cube, and n^3 is a perfect square. What is the number of digits of n?

    (A) 3 (B) 4 (C) 5 (D) 6 (E) 7

18. *(2003 AMC 10 A, ♯16)* What is the units digit of 13^{2003}?

    (A) 1 (B) 3 (C) 7 (D) 8 (E) 9

19. *(2008 AMC 10 A, ♯24)* Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$?

    (A) 0 (B) 2 (C) 4 (D) 6 (E) 8

20. *(2009 AMC 10 B, ♯21)* What is the remainder when $3^0 + 3^1 + 3^2 + \cdots + 3^{2009}$ is divided by 8?

    (A) 0 (B) 1 (C) 2 (D) 4 (E) 6

21. *(2006 AMC 10 B, ♯11)* What is the tens digit in the sum $7! + 8! + 9! + \cdots + 2006!$?

    (A) 1 (B) 3 (C) 4 (D) 6 (E) 9

(A) $\frac{4}{63}$ (B) $\frac{1}{8}$ (C) $\frac{8}{63}$ (D) $\frac{1}{6}$ (E) $\frac{2}{7}$

14. *(2004 AMC 10 B, ♯11)* Two eight-sided dice each have faces numbered 1 through 8. When the dice are rolled, each face has an equal probability of appearing on top. What is the probability that the product of the two top numbers is greater than their sum?

(A) $\frac{1}{2}$ (B) $\frac{47}{64}$ (C) $\frac{3}{4}$ (D) $\frac{55}{64}$ (E) $\frac{7}{8}$

15. *(2008 AMC 10 B, ♯16)* Two fair coins are to be tossed once. For each head that results, one fair die is to be rolled. What is the probability that the sum of the die rolls is odd? (Note that if no die is rolled, the sum is zero.)

(A) $\frac{3}{8}$ (B) $\frac{1}{2}$ (C) $\frac{43}{72}$ (D) $\frac{5}{8}$ (E) $\frac{2}{3}$

16. *(2008 AMC 10 B, ♯20)* The faces of a cubical die are marked with the numbers 1, 2, 2, 3, 3, and 4. The faces of a second cubical die are marked with the numbers 1, 3, 4, 5, 6, and 8. Both dice are thrown. What is the probability that the sum of the two top numbers will be 5, 7, or 9?

(A) $\frac{5}{18}$ (B) $\frac{7}{18}$ (C) $\frac{11}{18}$ (D) $\frac{3}{4}$ (E) $\frac{8}{9}$

17. *(2001 AMC 10, ♯19)* Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?

(A) 6 (B) 9 (C) 12 (D) 15 (E) 18

18. *(2003 AMC 10 A, ♯21)* Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

(A) 22 (B) 25 (C) 27 (D) 28 (E) 729

19. *(2002 AMC 10 B, ♯9)* Using the letters A, M, O, S, and U, we can form 120 five-letter "words". If these "words" are arranged in alphabetical order, then the "word" $USAMO$ occupies position:

(A) 112 (B) 113 (C) 114 (D) 115 (E) 116

20. *(2005 AMC 10 A, ♯9)* Three tiles are marked X and two other tiles are marked O. The five tiles are randomly arranged in a row. What is the probability that the arrangement reads $XOXOX$?

(A) $\frac{1}{12}$ (B) $\frac{1}{10}$ (C) $\frac{1}{6}$ (D) $\frac{1}{4}$ (E) $\frac{1}{3}$

21. *(2001 AMC 10, ♯25)* How many positive integers not exceeding 2001 are multiples of 3 or 4 but not 5?

(A) 768 (B) 801 (C) 934 (D) 1067 (E) 1167

22. *(2003 AMC 10 A, ♯8)* What is the probability that a randomly drawn positive factor of 60 is less than 7?

(A) $\frac{1}{10}$ (B) $\frac{1}{6}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

23. *(2003 AMC 10 A, ♯15)* What is the probability that an integer in the set $\{1, 2, 3, \ldots, 100\}$ is divisible by 2 and not divisible by 3?

(A) $\frac{1}{6}$ (B) $\frac{33}{100}$ (C) $\frac{17}{60}$ (D) $\frac{1}{2}$ (E) $\frac{18}{25}$

24. *(2006 AMC 10 A, ♯20)* Six distinct positive integers are randomly chosen between 1 and 2006, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 5?

(A) $\frac{1}{2}$ (B) $\frac{3}{5}$ (C) $\frac{2}{3}$ (D) $\frac{4}{5}$ (E) 1

25. *(2007 AMC 10 A, ♯16)* Integers a, b, c, and d, not necessarily distinct, are chosen independently and at random from 0 to 2007, inclusive. What is the probability that $ad - bc$ is even?

(A) $\frac{3}{8}$ (B) $\frac{7}{16}$ (C) $\frac{1}{2}$ (D) $\frac{9}{16}$ (E) $\frac{5}{8}$

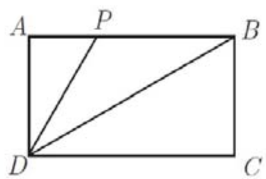
26. *(2001 AMC 10, ♯23)* A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

(A) $\frac{3}{10}$ (B) $\frac{2}{5}$ (C) $\frac{1}{2}$ (D) $\frac{3}{5}$ (E) $\frac{7}{10}$

4. *(2003 AMC 10 A, ♯17)* The number of inches in the perimeter of an equilateral triangle equals the number of square inches in the area of its circumscribed circle. What is the radius, in inches, of the circle?

(A) $\frac{3\sqrt{2}}{\pi}$ (B) $\frac{3\sqrt{3}}{\pi}$ (C) $\sqrt{3}$ (D) $\frac{6}{\pi}$ (E) $\sqrt{3}\pi$

5. *(2000 AMC 10, ♯7)* In rectangle $ABCD$, $AD = 1$, P is on $\overline{AB}$, and $\overline{DB}$ and $\overline{DP}$ trisect $\angle ADC$. What is the perimeter of $\triangle BDP$?

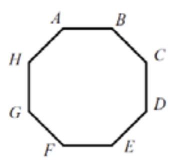


(A) $3+\frac{\sqrt{3}}{3}$ (B) $2+\frac{4\sqrt{3}}{3}$ (C) $2+2\sqrt{2}$ (D) $\frac{3+3\sqrt{5}}{2}$ (E) $2+\frac{5\sqrt{3}}{3}$

6. *(2001 AMC 10, ♯20)* A regular octagon is formed by cutting an isosceles right triangle from each of the corners of a square with sides of length 2000. What is the length of each side of the octagon?

(A) $\frac{2000}{3}$ (B) $2000(\sqrt{2}-1)$ (C) $2000(2-\sqrt{2})$ (D) 1000 (E) $1000\sqrt{2}$

7. *(2003 AMC 10 B, ♯23)* A regular octagon $ABCDEFGH$ has an area of one square unit. What is the area of rectangle $ABEF$?



(A) $1-\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{2}}{4}$ (C) $\sqrt{2}-1$ (D) $\frac{1}{2}$ (E) $\frac{1+\sqrt{2}}{4}$

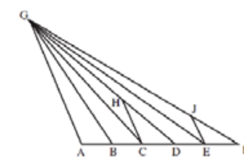
8. *(2005 AMC 10 A, ♯20)* An equiangular octagon has four sides of length 1 and four sides of length $\sqrt{2}/2$, arranged so that no two consecutive sides have the same length. What is the area of the octagon?

(A) $\frac{7}{2}$ (B) $\frac{7\sqrt{2}}{2}$ (C) $\frac{5+4\sqrt{2}}{2}$ (D) $\frac{4+5\sqrt{2}}{2}$ (E) 7

9. *(2002 AMC 10 B, ♯24)* Riders on a Ferris wheel travel in a circle in a vertical plane. A particular wheel has radius 20 feet and revolves at a constant rate of one revolution per minute. How many seconds does it take a rider to travel from the bottom of the wheel to a point 10 vertical feet above the bottom?

(A) 5 (B) 6 (C) 7.5 (D) 10 (E) 15

10. *(2002 AMC 10 A, ♯20)* Points A, B, C, D, E, and F lie, in that order, on $\overline{AF}$, dividing it into five segments, each of length 1. Point G is not on line AF. Point H lies on $\overline{GD}$, and point J lies on $\overline{GF}$. The line segments $\overline{HC}$, $\overline{JE}$, and $\overline{AG}$ are parallel. Find HC/JE.

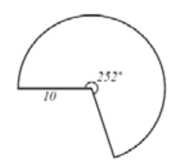


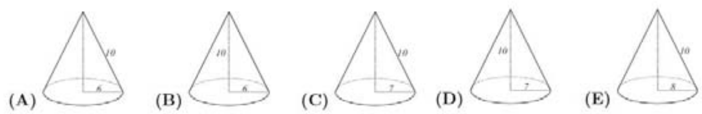
(A) 5/4 (B) 4/3 (C) 3/2 (D) 5/3 (E) 2

11. *(2002 AMC 10 A, ♯7)* If an arc of 45° on a circle A has the same length as an arc of 30° on circle B, then the ratio of the area of circle A to the area of circle B is:

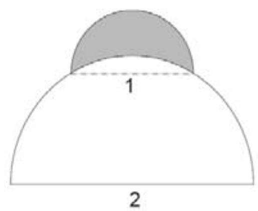
(A) $\frac{4}{9}$ (B) $\frac{2}{3}$ (C) $\frac{5}{6}$ (D) $\frac{3}{2}$ (E) $\frac{9}{4}$

12. *(2001 AMC 10, ♯17)* Which of the cones below can be formed from a 252° sector of a circle of radius 10 by aligning the two straight sides?



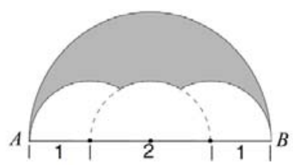


13. *(2003 AMC 10 A, ♯19)* A semicircle of diameter 1 sits at the top of a semicircle of diameter 2, as shown. The shaded area inside the smaller semicircle and outside the larger semicircle is called a *lune*. Determine the area of this lune.



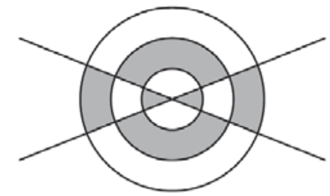
(A) $\frac{1}{6}\pi - \frac{\sqrt{3}}{4}$ (B) $\frac{\sqrt{3}}{4} - \frac{1}{12}\pi$ (C) $\frac{\sqrt{3}}{4} - \frac{1}{24}\pi$ (D) $\frac{\sqrt{3}}{4} + \frac{1}{24}\pi$ (E) $\frac{\sqrt{3}}{4} + \frac{1}{12}\pi$

14. *(2003 AMC 10 B, ♯19)* Three semicircles of radius 1 are constructed on diameter $\overline{AB}$ of a semicircle of radius 2. The centers of the small semicircles divide $\overline{AB}$ into four line segments of equal length, as shown. What is the area of the shaded region that lies within the large semicircle but outside the smaller semicircles?



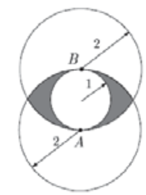
(A) $\pi - \sqrt{3}$ (B) $\pi - \sqrt{2}$ (C) $\frac{\pi + \sqrt{2}}{2}$ (D) $\frac{\pi + \sqrt{3}}{2}$ (E) $\frac{7}{6}\pi - \frac{\sqrt{3}}{2}$

15. *(2004 AMC 10 A, ♯21)* Two distinct lines pass through the center of three concentric circles of radii 3, 2, and 1. The area of the shaded region in the diagram is 8/13 of the area of the unshaded region. What is the radian measure of the acute angle formed by the two lines? (Note: π radians is 180 degrees.)



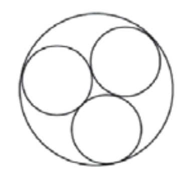
(A) $\frac{\pi}{8}$ (B) $\frac{\pi}{7}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{5}$ (E) $\frac{\pi}{4}$

16. *(2004 AMC 10 B, ♯25)* A circle of radius 1 is internally tangent to two circles of radius 2 at points A and B, where AB is a diameter of the smaller circle. What is the area of the region, shaded in the figure, that is outside the smaller circle and inside each of the two larger circles?



(A) $\frac{5}{3}\pi - 3\sqrt{2}$ (B) $\frac{5}{3}\pi - 2\sqrt{3}$ (C) $\frac{8}{3}\pi - 3\sqrt{3}$ (D) $\frac{8}{3}\pi - 3\sqrt{2}$ (E) $\frac{8}{3}\pi - 2\sqrt{3}$

17. *(2004 AMC 10 B, ♯16)* Three circles of radius 1 are externally tangent to each other and internally tangent to a larger circle. What is the radius of the large circle?



(A) $\frac{2+\sqrt{6}}{3}$ (B) 2 (C) $\frac{2+3\sqrt{2}}{3}$ (D) $\frac{3+2\sqrt{3}}{3}$ (E) $\frac{3+\sqrt{3}}{2}$