BAMO-8 Preparation
Berkeley Math Circle
February 19, 2013

This handout serves as a preparation for the BAMO-8. From 1999 to 2007 there was only one BAMO exam, and since 2008 there have been two: BAMO-8 and BAMO-12. The first three problems from previous BAMO exams roughly compare in difficulty level to the current BAMO-8.

Be sure to also check out the BAMO archives where you can find all the previous handouts, and also the BAMO prep materials in the BMC archives. Good luck!

1. **(BAMO, 1999, Problem 1)** Prove that among any 12 consecutive positive integers there is at least one which is smaller than the sum of its proper divisors. (The proper divisors of a positive integer n are all positive integers other than 1 and n which divide n. For example, the proper divisors of 14 are 2 and 7.)

2. **(BAMO, 2000, Problem 1)** Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1. (Two integers are relatively prime if they share no common positive divisor other than 1. For example, 22 and 15 are relatively prime, and thus $37 = 22 + 15$ represents the number 37 in the desired way.)

3. **(BAMO, 2002, Problem 1)** Let ABC be a right triangle with right angle at B. Let $ACDE$ be a square drawn exterior to triangle ABC. If M is the center of this square, find the measure of $\angle MBC$.

4. **(BAMO, 2005, Problem 1; modified)** An integer is called *formidable* if it can be written as a sum of distinct nonnegative integer powers of 4, and *successful* if it can be written as a sum of distinct nonnegative integer powers of 6. Can 2013 be written as a sum of a formidable number and a successful number? Prove your answer.

5. **(BAMO, 2004, Problem 2)** A given line passes through the center O of a circle. The line intersects the circle at points A and B. Point P lies in the exterior of the circle and does not lie on the line AB. Using only an unmarked straightedge, construct a line through P, perpendicular to the line AB. Give complete instructions for the construction and prove that it works.

6. **(BAMO, 2006, Problem 2; modified)** Since $24 = 3 + 5 + 7 + 9$, the number 24 can be written as the sum of at least two consecutive odd positive integers.

   (a) Can 2013 be written as the sum of at least two consecutive odd positive integers? If yes, give an example of how it can be done. If no, provide a proof why not.

   (b) How about 2006?

7. **(BAMO, 2007, Problem 2)** The points of the plane are colored in black and white so that whenever three vertices of a parallelogram are the same color, the fourth vertex is that color, too. Prove that all the points of the plane are the same color.

8. **(BAMO, 2009, Problem 2)** The *Fibonacci sequence* is the list of numbers that begins $1, 2, 3, 5, 8, 13$ and continues with each subsequent number being the sum of the previous two. Prove that for every positive integer n, when the first n elements of the Fibonacci sequence are alternately added and subtracted, the result is an element of the sequence or the negative of an element of the sequence. For example, when $n = 4$ we have $1 - 2 + 3 - 5 = 3$, and 3 is an element of the Fibonacci sequence.

9. **(BAMO, 2000, Problem 3)** Let $x_1, x_2, \ldots, x_n$ be positive numbers, with $n \geq 2$. Prove that

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\cdots\left(x_n + \frac{1}{x_n}\right) \geq \left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\cdots\left(x_{n-1} + \frac{1}{x_n}\right)\left(x_n + \frac{1}{x_1}\right).$$

10. **(BAMO, 2010, Problem 3)** Suppose a, b, c are real numbers such that $a + b \geq 0$, $b + c \geq 0$, and $c + a \geq 0$.

Prove that

$$a + b + c \geq \frac{|a| + |b| + |c|}{3}.$$

(Note: $|x|$ is called the *absolute value* of x and is defined as follows. If $x \geq 0$, then $|x| = x$; and if $x < 0$ then $|x| = -x$. For example, $|6| = 6$, $|0| = 0$ and $|-6| = 6$.)

11. (BAMO, 2007, Problem 4) Let N be the number of ordered pairs (x, y) of integers such that

$$x^2 + xy + y^2 \leq 2007.$$

Remember, integers may be positive, negative, or zero!

(a) Prove that N is odd.

(b) Prove that N is not divisible by 3.

12. (BAMO, 2010, Problem 4) Place eight rooks on a standard 8×8 chessboard so that no two are in the same row or column. With the standard rules of chess, this means that no two rooks are attacking each other. Now paint 27 of the remaining squares (not currently occupied by rooks) red. Prove that no matter how the rooks are arranged and which set of 27 squares are painted, it is always possible to move some or all of the rooks so that:

- All the rooks are still on unpainted squares.
- The rooks are still not attacking each other (no two are in the same row or same column).
- At least one formerly empty square now has a rook on it; that is, the rooks are not on the same 8 squares as before.