Computer Programming in the 18th Century
(OK, really, finite differences)

Berkeley Math Circle
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1 Warm-up Problems

These are *usually* in order of difficulty – but not today! The last few are about polynomials, and are probably the most relevant to what we'll be doing in class, but you may work on any that you like.

Problem 1 Can you write the numbers $1, 2, 3, \ldots, 100$ in a row such that the difference between any two adjacent numbers is at least 50?

Problem 2 There are 25 people sitting around a circular table. Each person has two cards. Each card has one of the numbers $1, 2, 3, \ldots, 24, 25$ on it, and each number appears on exactly two cards. At a signal, each person passes the card he or she holds with the smaller number to his or her neighbor on the right. Prove that, if this process is repeated indefinitely, sooner or later, one of the players will have two cards with the same number.

Problem 3 Thirty empty chairs sit in a row. From time to time, someone comes and sits in an empty chair. If anyone is sitting in a chair immediately next to the new arrival, then one such person stands up and leaves. (that is, if the new arrival sits in an empty chair directly between two people already sitting, just one of those two people leave). What is the maximum possible number of people who can be sitting on the chairs if this process continues indefinitely?

Problem 4 Find a polynomial of smallest possible degree for which $p(x)$ has $p(0) = 2$, $p(1) = 1$, $p(2) = -2$ and $p(3) = -6$. What is $p(4)$? (or you can try a simpler version of this problem in which we seek the polynomial $q(x)$ of smallest degree for whcih $q(0) = 2$, $q(1) = -1$, and $q(2) = 0$.)

Problem 5 Find a polynomial of degree at most 3 whose graph goes through the points $(-1, 5)$, $(3, 14)$, $(6, 2)$, $(11, 3)$.

Problem 6 (harder, but involves some of the same ideas) If $P(x)$ is a polynomial of degree 2010, and $P(1) = 1$, $P(2) = 1/2$, $P(3) = 1/3$, $P(4) = 1/4$, ... , $P(2011) = 1/2011$, find $P(2012)$.

2 Introduction – Digression – next number in a sequence, pattern in a sequence

Problem 7 For each the following sequences, try to analyze

- What is "the" next number (or two) in "the" sequence?
- What is ~~the~~ **a** pattern that characterizes your sequence? (What types of descriptions count as a pattern?)
- Better still, Find as many patterns as you can describing the sequence.
- For each pattern, can you find other sequences that meet the same pattern? Can you characterize (in some way) the family of sequences?

(A) $3, 7, 11, 15, 19, \ldots$

(B) $13, 6, -1, -8, -15, -22, \ldots$

(C) $-1, 0, 1, 4, 9, 16, \ldots$

(D) $0, 4, 11, 21, 34, \ldots$

(E) $1, 1, 3, 13, 37, 81, \dots$

(F) $-13, 5, 9, 5, -1, -3, 5, 29, 75, 149, 257, \dots$

(G) $1, 2, 4, 8, 16, 32, \dots$

(H) $1, 1, 2, 3, 5, 8, 13, 21, 34, \dots\dots$

Problem 8 Once we have introduced finite differences, let's go back and apply them to our example functions. Let's also try this process with the following functions: $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^4$. What happens? We should try to come up with some hypotheses, and maybe gather some evidence. Can you think of another family of polynomials that might be worth trying?

Problem 9 Looking at the array of differences you get for x^2 and x^3, and compare them to the array of differences for $3x^2$ and $2x^3$. Then compare those to the array differences for $3x^2 - 2x^3$. See any patterns?

Problem 10 the general problem if I know values on one diagonal $d_0, d_1, d_2, \dots d_n$ (and also that the rows below d_n is entirely 0)

$$\begin{array}{ccccccccc}
d_0 & ? & ? & ? & ? & ? & \dots & & \\
 & d_1 & ? & ? & ? & ? & ? & \dots & \\
 & & d_2 & ? & ? & ? & ? & \dots & \\
 & & & d_3 & ? & ? & ? & ? & \dots \\
 & & & & 0 & 0 & 0 & 0 & \dots
\end{array}$$

Can I determine the sequence on the top row? Can I express it in a formula in terms of $d_0, d_1, \dots, d_n$?

Problem 11 If the previous problem is too hard, can we solve it in some special cases? What if $d_0 = d_1 = d_2 = \dots = d_{n-1} = 0$, and onlye $d_n = 1$?

3 A useful family of polynomials

You may also know the expression

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n(n-1)\cdots(n-m+1)}{m!}$$

in connection with binomial coefficients and Pascal's triangle, but we can also consider them as polynomials in their own right:

$$\binom{x}{m} = \frac{x(x-1)\cdots(x-m+1)}{m!}$$

These are interesting polynomials! Let's plot a few values of them and see what their finite differences look like.

4 A recurrence Relation for polynomials

Finite differences give one way to show that:

$$p(x+n) = \binom{n}{1}p(x+n-1) - \binom{n}{2}p(x+n-2) + \dots + (-1)^{n-1}p(x)$$

for any polynomial of degree *less* than n. (Compare this to what we noticed for arithmetic sequences).

5 Problems that naturally lead to finite differences

Problem 12 Any problem where the sequence of solutions satisfies $a_{n+1} = a_n + P(n)$ where $P(n)$ is a polynomial.

- $a_{n+1} = a_n + k$
- $a_{n+1} = a_n + n$

We might need a starting point a_0 or a_1.

Problem 13 In particular, many summations

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

can be evaluated with this approach, since $S_{n+1} - S_n = \ldots$.

Problem 14 Can we evaluate:

1. $\sum_{k=1}^{n} k, \sum_{k=1}^{n} k^2, \sum_{k=1}^{n} k^3$
2. $\sum_{k=1}^{n} k \cdot (k+3)$
3. $\sum_{k=1}^{n} k^{\underline{3}}, \sum_{k=1}^{n} k^{\underline{4}}$
4. $\sum_{k=1}^{n} \sum_{j=1}^{k} j^2$ (This one cameup recently in a problem I heard from Josh Zucker: Given an $n \times n$ square grid of points, how many squares can you make by connecting the points *not including* those squares whose sides are parallel to the sides of the grid?)

Problem 15 (Common) Into how many pieces can a pizza be divided by n straight vertical cuts? (Assume the pizza is essentially 2-dimensional – also convex. And no moving the pieces around between the cuts.)

Problem 16 Into how many pieces can a cake be cut with n straight cuts (not necessarily vertical! The point is that the cake has thickness, so now the shape is 3-dimensional and the cuts are not lines, but planes!)

Problem 17 (AIME 1992) For any sequence of real numbers $A = (a_1, a_2, a_3, \ldots)$, define ΔA to be the sequence $(a_2 - a1, a_3 - a_2, a_4 - a_3, \ldots)$, whose nth term is $a_{n+1} - a_n$. Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1 and that $a_{19} = a_{92} = 0$. Find a_1.

Problem 18 (More repertoire method than finite differences) The polynomial equation $x^2 - x - 1 = 0$ has the two solutions $\phi = \frac{1+\sqrt{5}}{2} = 1.61803399\ldots$ and $\Phi = -0.61803399\ldots$. The recurrence relation $a_{n+1} = a_n + a_{n-1}$ has many solutions, the most famous being the fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. Show that the geometric sequences $\phi^1, \phi^2, \phi^3, \ldots$ and $\Phi^1, \Phi^2, \Phi^3, \ldots$ satisfy the same recurrence relation. Verify that, if you can find a and b fo rwhich $1 = a\phi^1 + b\Phi^1$ and $1 = a\phi^2 + b\Phi^2$, then the nth Fibonacci number must be $a\phi^n + b\Phi^n$.

6 Ok, the Computer Programming in the 18th Century part

A brief outline of the history

- People were doing sophisticated mathematics and computations before there were electronic calculators and computers to evalute the functions for them.
- There are ways to compute logarithms, trig functions, square roots, etc, by hand using only arithmetic operations, but they are often complex, messy, somewhat tedious and not practical for everyday use.
- Premade mathematical tables of values were published and very commonly used by everyone with a need for computation (engineers, machinists, navigators, surveyors, architects, designers and others).
- Producing a large table of mathematical values without computing equipment involved teams of people and careful planning by mathematicians to break the task into easy (if tedious) steps.
- The first useful fact is that any "smooth" function can be approximated (at least in small interval) by a polynomial. It's much easier to evaluate a polynomial than it is a logarithm or trig function – you just have to know how to add and multiply!
- But even that is too messy do thousands of times – you don't want to have to multiply numbers with 10 decimal places together over and over. The second useful fact is that you can evaluate a polynomial at a series of equally spaced points using only addition with the method of finite differences.
- So, to make a table of values for a function with a certain step-size:
  - A high-level mathematician would find a good polynomial to approximate a given function over a particular interval. Competing goals: want as many decimal places of accuracy as possible, but with as low a degree polynomial and over as large an interval as possible.
  - a medium-level mathematician, given the polynomial and starting point from the high-level mathematician, would compute the starting values of the finite difference for the given step size
  - teams of low-level "computers", who knew how to add and could follow instructions, would then crank away computing the values of the polynomial at successive steps.
- This process worked, but many errors crept in, both from the computations themselves or when printers typeset the handwritten manuscripts. Elaborate systems of error checking and proofreading were developed, but some errors remained.
- Charles Babbage's Difference Engine – you can see one of only two working models in the world at the Computer History Museum in Mountain View! – designed in the 1820s-1840s, would have automated the process, replacing the "computers" and the typesetters with clockwork-like gears. The high-level and medium-level mathematicians still had to do their work to set things up, but once that was done, the rest would happen without error by simply turning a crank.

7 Examples

7.1 Square Root Table

How would you compute $\sqrt{23.2}$ to eight decimal places? We can explore various methods.

And, really, our goal is to find a way to compute hundreds or thousands of values of $\sqrt{x}$, say $x = 22, 22.01, 22.02, 22.03, \ldots, 23.99, 24.00$. We don't mind doing a little work up front to get the process started, but once it starts, we'd like to keep the number and complexity of computations for each additional value in our table as low as possible.

What I'd really like is

- A polynomial whose graph is very close to the graph of $\sqrt{x}$ in the interval near my point.

  - linear interpolation from $(16, 4)$ to $(25, 5)$:
  - linear interpolation from $(20.25, 4.5)$ to $(25, 5)$.
  - quadratic interpolation through $(16, 4)$, $(20.25, 4)$, $(25, 5)$
  - fifth degree polynomial through $(20.25, 4.5)$, $(21.16, 4.6)$, $(22.09, 4.7)$, $(23.04, 4.8)$, $(24.01, 4.9)$, $(20.25, 4.5)$
- A way to turn that polynomial into finite differences, so my computers can more easily do the calculations. (Start with $g(x) = f(22 + \frac{x}{100})$)

Computing the polynomial and differences to 10 decimal places, I get:

$$\begin{array}{cccccccc}
4.6904157598 & & ? & & ? & & ? & \dots \\
& 0.0010658825 & & ? & & ? & & ?\dots \\
& & -0.0000002421 & & ? & & ? & \dots \\
& & & 0.0000000002 & & ? & & ?\dots
\end{array}$$

And the fifth difference, to ten decimal places, is 0!

7.2 Tangent table

Suppose we want to make a table of values of the tangent function, and today we're in the vicinity of 74 degrees. Our publisher wants the step size to be 1-minute (one sixtieth of a degree).

Our high-level mathematician tells us that the polynomial

$$P(x) = 3.6058835 + 0.244388 \cdot (x - 74.5) + 0.015394 \cdot (x - 74.5)^2 + 0.000993 \cdot (x - 74.5)^3$$

approximates $\tan x^\circ$ to within 0.000001 (10^{-6}) for all x between 74 and 75. (and within 10^{-4} for all x between 73.4 and 73.6. (and within 0.02 for all x between 70 and 78). (In "real life," we'd use a higher degree polynomial that had at least this much accuracy over a somewhat larger interval.)

This is already sort of useful, if you wanted to calculate the tangent of 74 degrees, 6 minutes, you could evaluate $P(74.1) = 3.6058835 + 0.244388 \cdot (-.4) + 0.015394 \cdot (-.4)^2 + 0.000993 \cdot (-.4)^3$, which you *could* do by hand if you had to. But we're not yet ready for our assembly line.

Our medium-level mathematician then takes this polynomial, and knowing that we want to start at 74 degrees and go up by steps of 1 minute (that is, one-sixtieth of a degree), calculates that our difference equation should start with:

$$\begin{array}{cccccccc}
3.487413875 & & ? & & ? & & ? & \dots \\
& 0.003832846 & & ? & & ? & & ?\dots \\
& & 0.000007752 & & ? & & ? & \dots \\
& & & 0.000000028 & & ? & & ?\dots
\end{array}$$

And now, our calculators would start doing their additions and the next sixty numbers in the top row of their computations would give us all the tangents we seek to within the desired accuracy.