Curious Number Systems

Berkeley Math Circle

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■ Base-Two

1. Investigate the claim that every integer has a unique representation as a sum of powers of 2.

2. Use the *Russian Peasant Method* to multiply
   a) 16×15 b) 14×26 c) 13×28

3. a) Show that $7 = 111_2$ directly.
   b) Verify that $15 = 1111_2$ by using part a).
   c) Verify that $31 = 11\,111_2$ by using b).
   d) How does this show that $2^n - 1 = 111 \cdots 11_2$?

4. Use induction to prove the same fact, namely that $2^{n+1} - 1 = 2^n + 2^{n-1} + \cdots + 2 + 1$

5. Among 50 bottles of soda, there is one containing a deadly poison.
   After 24 hours the poison causes complete paralysis.
   You have 6 lab rats. Devise a strategy to determine which bottle contains the poison.
   What is the least amount of time in which you can do this?

6. Examine your set of six magic cards. Given any subset of the six, find a quick method to determine what number appears uniquely on those cards and not the others.

7. Investigate the claim that every integer has a unique representation as a sum of single Fibonacci numbers. Use this fact to convert kilometers to miles or miles to kilometers.

■ Division After the Decimal

1. Practice the division algorithm; show that $\frac{1}{7} = .\overline{142\,857}$.

   Repeat with $\frac{2}{7}$, $\frac{3}{7}$. Make a conjecture about $\frac{4}{7}$, $\frac{5}{7}$ and $\frac{6}{7}$.

2. The pictures below demonstrate that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1$, and
   $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots + \frac{1}{4^n} + \cdots = \frac{1}{3}$ and $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n} + \cdots = \frac{1}{2}$.

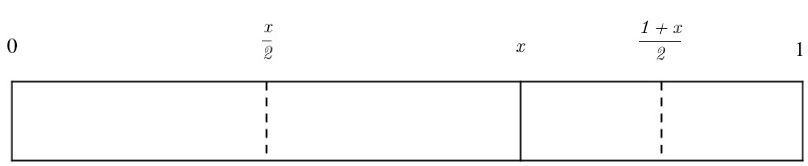
3. Verify that 1 = .9999 .... and investigate the base-two counterpart of this statement.

4. Use the division algorithm to verify that $\frac{1}{3} = (.010101\ ...)_2$ and $1 = (.111\ ...)_2$
   Note that this is a counterpart to problem 4.

5. Verify that $\frac{1}{5} = (.001100110011\ ...)_2$ two ways.

   Use this to compute $\frac{2}{5}$ and $\frac{3}{5}$ *without* using the division algorithm.

6. With a mark at x on a strip of paper, we can fold to meet x from either the right or the left.
   Verify that the right crease ends up at $\frac{1+x}{2}$ and the left crease is at $\frac{x}{2}$.



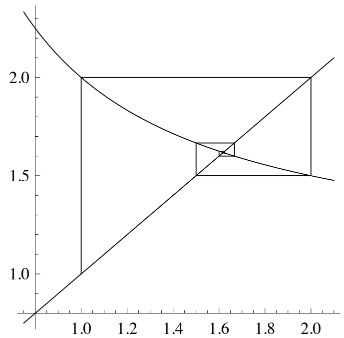
7. Use the previous observation to show that a sequence of left-right-left-right-left... folds result (ultimately) in a crease at a thirds mark.
   We saw that right-right-left-left-... results in a crease at the one-fifth mark. What sequence results in a crease at the one-seventh mark.

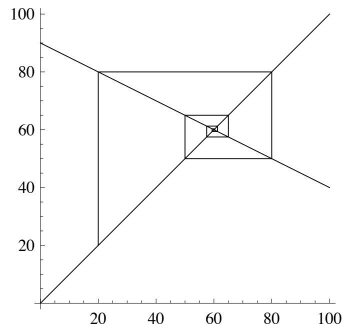
8. If we think of x as a decimal number base-2, then the left-fold pushes a *zero* onto the front, resulting in a crease at $(.0\,x)_2$; a right-fold pushes a *one* onto the front, resulting in a crease at $(.1\,x)_2$.

9. Use problem 2 to show that a sequence of left-right-left-right-left... folds result (ultimately) in a crease at a thirds mark.
   What sequence of folds will result in a crease at a fifths mark? A seventh mark?

■ Another Experiment

1. Let $f(x) = 1 + \frac{1}{x}$. What is $f(1)$? $f(f(1))$? $f(f(f(1)))$?
   In other words, we repeat the operations "take the reciprocal and add one."
   Find the pattern; what is the result after any number of repetitions?



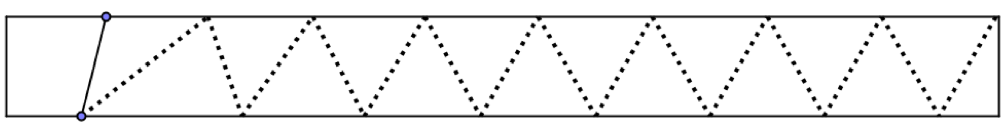


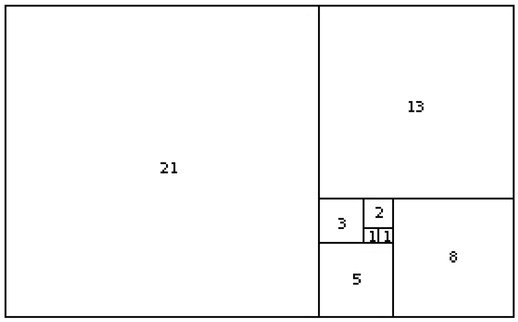
■ Folding Triangles

5. Make any fold at the left-hand side of your strip.
   Next, fold the bottom edge up to meet your crease;
   Next, fold the top edge down to meet your crease;
   Next, fold the bottom edge up to meet your crease;
   Next, fold the top edge down to meet your crease.
   Repeat until your creases fill the strip.

   What do you notice?

   Explore a shortcut: turn the strip over so that the bottom edge is repeatedly folded to the crease.




21
13
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2
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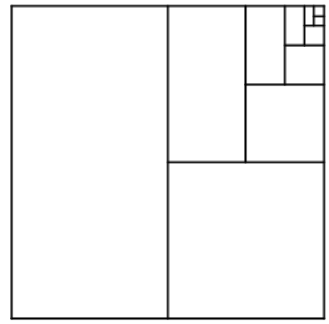

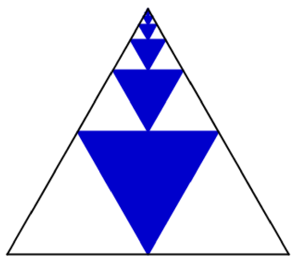

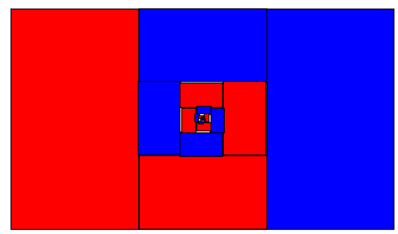

1	2	**3**	4	**5**	6	**7**	8
9	10	**11**	12	**13**	14	**15**	16
17	18	**19**	20	**21**	22	**23**	24
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33	34	**35**	36	**37**	38	**39**	40
41	42	**43**	44	**45**	46	**47**	48
49	50	**51**	52	**53**	54	**55**	56
57	58	**59**	60	**61**	62	**63**	64

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