Plane Geometry:

Reflection Across a Line Does It!

Berkeley Math Circle – Beginners

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Note: This handout is designed for a series of 4-sessions. Today we shall start talking and thinking about the main two problems. Try to understand what the problems say and draw pictures for them as best as you can. You are not expected to be able to solve the problems on your own, but rather keep these problems in mind and think about them for a while. In the future, we will revisit this topic and finish the whole handout. Some of the problems in this handout are from "Mathematical Olympiads", part II by Stoyan Budurov and Dimo Serafimov, State Publishing Company "Narodna Prosveta", Sofia, 1985.

1. Overarching Problems

(1) **(Three Squares)** Three identical squares with bases AM, MH, and HB are put next to each other to form a rectangle $ABCD$. Find the sum of the angles $\angle AMD + \angle AHD + \angle ABD$ and prove that your answer is correct.

(2) **(Farmer and Cow)** During a hot summer day, a farmer and a cow find themselves on the same side of a river. The farmer is 2 km from the river and the cow is 6 km from the river. If each of them would walk straight to the river, they would find themselves 4 km from each other. Unfortunately, the cow has broken its leg and cannot walk. The farmer needs to get to the river, dip his bucket there, and take the water to the cow. To which point on the river should the farmer walk so that his total walk to the river and then to the cow is as short as possible? Prove your claim.

2. Extensions

(3) (**Shortest Broken Line**) Two lines p_1 and p_2 intersect. Two points A and B lie in the acute angle formed by the lines. Find a point C on p_1 and a point D on p_2 so that the broken line $ADBCA$ has the smallest possible length. Prove that the points you have found indeed yield this smallest possible length.

(4) (**Locating Angle Bisector**) Two points A and B and a line l are given so that the line intersects segment AB (neither A nor B lies on l). Find point C on l so that the angle bisector of $\angle ACB$ lies on l. Prove that your construction is correct.

3. Preparation with Reflections: How to Draw and Basic Properties

(5) **(Parallel axes of reflection)** Line g_1 and g_2 are parallel and point M does not lie on any of them.

(a) Find the reflection M_1 of M across g_1 and the reflection M_2 of M across g_2.

(b) If d is the (shortest) distance between g_1 and g_2, prove that $|M_1M_2| = 2d$.
Consider the case when M is between the two parallel lines and the case when it is not between them.

(6) **(Reflections on a circle)** Two lines g_1 and g_2 intersect in point O, and point M does not lie on any of the lines. Let M_1 be the reflection of M across g_1, M_2 the reflection of M_1 across g_2, and M_3 the reflection of M across g_2. Prove that

(a) points M, M_1, M_2, and M_3 lie on a circle with center O.

(b) the line determined by O and the midpoint of M_2M_3 is perpendicular to M_2M_3.

Theorem 1. *Segments preserve their length under reflection across a line. In other words, if we reflect segment AB across line l to segment A_1B_1, then $|AB| = |A_1B_1|$.*

Theorem 2. *In $\triangle ABC$ the segment connecting C to the midpoint M of side AB is perpendicular to AB if and only if $\triangle ABC$ is isosceles with $|AC| = |BC|$.*

(7) **(Reflections and angles)** Two lines g_1 and g_2 intersect in point O, and point M does not lie on any of the lines.

(a) Draw the reflection M_1 of M across g_1 and the reflection M_2 of M_1 across g_2.

(b) In case O does not lie on line MM_2, prove that $\angle M_2MO = \angle MM_2O$.

(c) If $|OM| = 9\,\text{cm}$ and $\angle MOM_2 = 120°$, find the distance from O to line MM_2.

Theorem 3. *Angles are preserve their measure under reflection across a line. In other words, if three points A, B and C are reflected across line l to points A_1, B_1 and C_1, then $\angle ABC = \angle A_1B_1C_1$.*

Theorem 4. *In a 30°-60°-90° triangle, the shorter leg is half of the hypothenuse. (Note: The shorter leg lies against the 30° angle. Why?)*

(8) **(Reflections and rectangle)** Lines m and n are mutually perpendicular and intersect in point A. Point B does not lie on any of these lines but is in their plane. Point B_1 is the reflection of B across m, point B_2 is the reflection of B_1 across n, and point B_3 is the reflection of B_2 across m.

(a) Draw points B_1, B_2, and B_3.

(b) Prove that points B, B_1, B_2, and B_3 are the vertices of a rectangle.

(c) If point B is at a distance 1.5 cm from line n and segment AB makes an angle of 30° with line n, find the length of AB.

Theorem 5. *A quadrilateral is a rectangle if and only if its diagonals intersect each other in a point equidistant from all four vertices. In other words, a quadrilateral $ABCD$ is a rectangle exactly when its diagonals AC and BD intersect in point O such that $|OA| = |OB| = |OC| = |OD|$.*

(9) **(Reflections and angle bisector)** In $\triangle ABC$ the angle bisector of $\angle BAC$ lies on line g_1, while line g_2 is perpendicular to g_1 and passes through point A.

(a) Prove that the angle bisector of the supplementary angle to $\angle BAC$ lies on g_2.

(b) If B_1 and B_2 are the reflections of B across line g_1 and g_2, respectively, prove that B_1 and B_2 lie on line AC.

(c) Find the length of segment BB_1, given $|AB| = 2.5\,\text{cm}$ and $\angle AB_2B = 30°$.

Theorem 6. *Let α and β be two* supplementary *angles, i.e., α and β share one ray, and their other rays are opposite to each other. Then the* angle bisectors *of α and β are* perpendicular *to each other.*

(10) **(Reflecting a whole line)** Given are lines a, b, and s. On line b draw point A_1 so that its reflection across s lies on a. After that locate this reflection of A_1 on line a.

Theorem 7. *The reflection of a line* b *across a line* s *is another line* b_1. *Moreover, either both lines* b *and* b_1 *make the same angle with* s, *or they are both parallel to* s.

4. Related Exercises for Work at the Math Circle and at Home

(11) What is *reflection across line s*? Describe in words.

(12) How do we draw reflections of points across s? Draw several examples. Are there any special cases we need to consider? What instruments do we need to draw such reflections precisely? (*Hint:* The answer here includes, among other things, being able to drop a perpendicular from a point to a line.)

(13) There are essentially *three* cases in Problem 5: two *outside* cases and one *inside* case. Why does the problem ask you to consider only two cases? Explain.

(14) Theorem 2 has two directions:

(a) You assume that $|AC| = |BC|$ and then prove that CM is perpendicular to AB; the latter is written as $CM \perp AB$.

(b) You assume that $CM \perp AB$ and then prove that $|AC| = |BC|$.

Prove each directions. Of course, you are allowed to use and obliged to state which criteria for congruent triangles you use.

(15) What is a *circle*? Describe carefully and precisely, in words.

(16) Given segment AB, what instruments do you need to find precisely the midpoint M of AB? Describe the construction step by step. (*Hint:* The previous exercise on Theorem 2 essentially gives away the idea: you need to construct an extra special point C.)

(17) Prove Theorem 1. Consider separately four cases, depending on whether segment AB and line l intersect and whether they are perpendicular to each other.

(18) Prove Theorem 3. For a full and formal solution, you may need to consider a bunch of cases dependending on the relative position of $\angle ABC$ with respect to (wrt) line l. Just concentrate on a generic case, e.g., all three points are on one side of l and none of the sides of $\triangle ABC$ is perpendicular to l.

(19) **(For the die-hards)** Prove Theorem 4. (*Hint:* One way is to pick the midpoint of the hypothenuse, show that it is the center of the circle passing through the vertices of the triangle, and see why this introduces a new equilateral triangle in your picture.)

(20) Prove Theorem 5. Note that it requires the proof of two directions.

(21) Draw two supplementary angles neither of which is right. Draw their angle bisectors and notice that they seem perpendicular to each other. Prove Theorem 6.

(22) What is an *angle bisector* of $\angle ABC$? Describe carefully in words. Assuming we are forbidden to use a protractor, what *other* instruments do we need to draw the angle bisector of angle? Describe the construction step by step.

(23) Can you think of *two different* solutions to Problem 10?

(24) Investigate what happens in the special positions for the lines a, b and s in Problem 10. Does the solution to Problem 10 change? Why or why not? Explain.

(25) Find out and describe all positions of the lines a, b and s in which Problem 10 has **no** solution. Explain in words what goes wrong and why there is no solution. (*Hint:* One way to look at it is that a certain isosceles triangle will cause trouble. There is another way to discribe this troublesome situation, using the word *reflection*.)

(26) **(For the experienced)** Prove Theorem 7.

(27) **(For the die-hard explorers)** Suppose in Problem 5 we keep on reflecting:

$$M \xrightarrow{g_1} M_1 \xrightarrow{g_2} M_2 \xrightarrow{g_1} M_3 \xrightarrow{g_2} M_4 \xrightarrow{g_1} \cdots .$$

What is the distance $|MM_{2010}|$? How about $|MM_{2011}|$? What additional initial data do you need to know to determine these distances? Explain.

(28) What does it mean that 3 lines are in *general* position?

(29) What special positions for 3 lines can you think of?

(30) What happens if we reflect *all* points of a line b across s? What figure do you get? Do we need to reflect *all* points of b across s in order to find where b goes?

5. Further Applications of Reflections

(31) **(Minimal perimeter, again!)** Given $\triangle ABC$, on the ray opposite to ray $\overrightarrow{CA}$ take a point B_1 so that $|CB_1| = |CB|$. Prove that

(a) Point B_1 is the reflection of B across the angle bisector l of the exterior angle of the triangle at vertex C.

(b) If D is an arbitrary point on l different from C, the perimeter of $\triangle ABD$ is bigger than the perimeter of $\triangle ABC$.

(32) **(Find the perimeter!)** In $\triangle ABC$, $|AC| = |BC|$ and $|AB| = 10\,\text{cm}$. Through the midpoint D of AC we draw a line perpendicular to AC. This line intersects BC in point E. The perimeter of $\triangle ABC$ is $40\,\text{cm}$. Find the perimeter of $\triangle ABE$.

(33) **(Medians and angle bisector)** In isosceles $\triangle ABC$ ($|AC| = |BC|$) the medians to the congruent sides intersect in point D. Prove that segment DC divides the angle of the triangle at C into two equal parts.

(34) **(A fifth criteria for triangle congruence)** Prove that if in a given triangle one of its sides, an angle adjacent to that side, and the sum of the other two sides of the triangle are correspondingly congruent to the same elements in a second triangle, then the two triangles are congruent.