PRIMES AND PROOFS, 4/2/2013

1) Can you find a prime factorization for the following numbers? Are there other possible prime factorizations?

a) 2940
b) 4568
c) 571
d) 4572
e) 3690

2) Let's practice a proof by induction. We will show that $1 + 2 + \cdots + n = \frac{n^2+n}{2}$. Note that there is more than one way to prove this, but we'll try the induction way.

a) Check the base case of $n = 1$.

b) What should the "inductive hypothesis" be?

c) What do we want to show given the inductive hypothesis?

d) Show it!

3) Suppose we are given three positive whole numbers (integers) a, b, c. Suppose that $a \times b$ is divisible by c but b and c share no factors bigger than 1. In this problem, we will think about whether it is then true that a must be divisible by c.

a) First logical thing to do is to try this on a few examples. Suppose $a = 12$, $b = 10$, $c = 3$. Is it true that $a \times b$ is divisible by c? IS it true that b and c share no common factors bigger than 1? Is it true that a is divisible by c?

b) Maybe in part (a) we just got lucky. Let's try another example. Suppose $a = 30$, $b = 17$, $c = 5$. Is it true that $a \times b$ is divisible by c? IS it true that b and c share no common factors bigger than 1? Is it true that a is divisible by c?

c) Now let's try to prove that it is always true. In other words, we will prove the following theorem: Let a, b, c be positive whole numbers. Suppose that $a \times b$ is divisible by c but b and c share no factors bigger than 1. Then a must be divisible by c.

STEP 1: What is the greatest common divisor of $a \times b$ and $a \times c$?

STEP 2: Is the greatest common divisor of $a \times b$ and $a \times c$ divisible by c? Why or why not?

Why does this finish the proof?

4) We will now prove a "corollary" of the theorem we proved in the last problem. Again, suppose that a and b are positive whole numbers. We will show that if p is a prime number and $a \times b$ is divisible by p then a is divisible by p or b is divisible by p (or both).

a) If we have that both p and a and p and b share some common factors bigger than 1 then we are done. Why?

b) Suppose that p shares no common factor with a. How do we know that then b must be divisible by p? Hint: think about what we proved in the last problem.

c) Suppose that p shares no common factor with b. How do we know that then a must be divisible by p? Does this finish off the proof?

Optional Problem: We say that a positive fraction is written in *reduced form* if it is written as $\frac{a}{b}$ where a and b are positive whole numbers that share no factor bigger than 1. Is it true that there is only one way to write any given positive fraction in reduced form? Prove it. You might need the theorem you proved in problem 3.

5) Write out a list of prime numbers less than or equal to 45 which can be written as a sum of two squares of whole numbers. Try to make sure your list is complete.

6) Write out a list of whole numbers less than or equal to 45 which can be written as a sum of two squares of whole numbers. Try to make sure your list is complete.