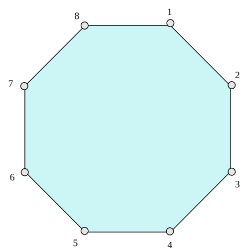
Polygons, Triangulations, Frieze Patterns

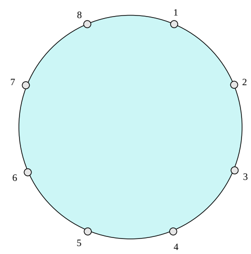
Berkeley Math Circle
Karin Baur, November 6, 2012

1. Triangulations

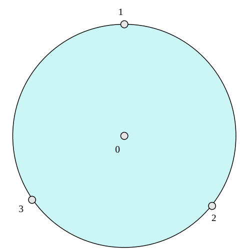
I. Definitions. A **polygon**: a regular figure with n vertices, in the plane, with n (straight) sides of equal length, with equal angles at all vertices. A **diagonal** in a polygon is a straigth line connecting two vertices (not neighbors).
Consider different figures:



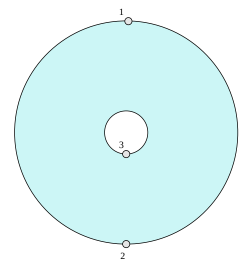
polygon



disc



punctured disc



annulus

1: Use diagonals, arcs, to dissect these figures.

2: What is a triangulation? How to define it?

3: How to triangulate a punctured disc? An annulus?

4: What shapes do we get from triangulations?

5: How does all this look on a torus (doughnut) with one point?

II. Counting diagonals/arcs, triangulations. A number of counting questions:
6: How many diagonals/arcs can you find in these figures?

7: How many diagonals/arcs do your triangulations use, how many regions?

Solution to 7, number of arcs: $6g + p + 3q - 3(2 - b)$
p points, q punctures, b boundaries, g handles (genus).
Question What are p, q and b for polygon, punctured disc, annulus, doughnut?

8: How many ways are there to triangulate a square, pentagon, hexagon, polygon?

Answer tor 8: square: 2, pentagon: 5, hexagon: ?, heptagon: ? $\rightsquigarrow \frac{(2n)!}{(n+1)!n!}$
(polygon with $n + 2$ vertices).

9: Question 8 for annulus, torus?

Hatcher (1991): any two triangulations of such a figure are related by a sequence of flips.

In particular: look at square, at pentagon. Graph for hexagon:

Triangulations of a polygon with $n + 3$ vertices $\rightsquigarrow$ n-regular graph.

2. Frieze patterns

10: What is a frieze pattern? Find one.

Today: Conway-Coxeter frieze patterns of non-negative integer numbers. They obey certain rules. "Diamonds".

Example. Consider the following band of numbers in the plane:

$$\cdots\quad\begin{array}{ccccccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 \\
 & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & \\
4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}\quad\cdots$$

This pattern has order 6 (for 5 rows of positive entries).
11: Can you find rules for this? Symmetries?

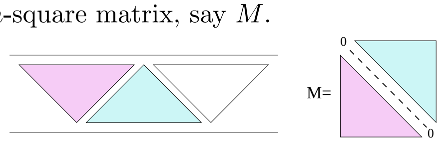
12: Can we vary entries?

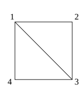
Conway-Coxeter (1973): such patterns arise (exactly) from triangulations of polygons. n-gon $\rightsquigarrow$ order n pattern.

13: Further question: How many differnt kinds of frieze patterns can you come up with? (You should find 7).

Determinant. A square matrix: a grid with $n \times n$ entries (numbers). The determinant, det, is a function on these: $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, $\det \begin{pmatrix} 1 & -2 & 0 \\ 4 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 2$

Determinant of a frieze pattern. The triangular regions of a frieze patterns of order n form an $n \times n$-square matrix, say M.



14: Draw the frieze pattern of 

and write its matrix.

Broline-Crowe-Isaacs (1974):

$$\det M = -(-2)^{n-2}$$

Proof: polygon geometry, combinatorics (matchings), linear algebra.