Algebra Toolkit

1 *Rules of Thumb*.

- Make sure that you can prove all formulas you use. This is even better than memorizing the formulas. Although it is best to memorize, as well.
- Strive for elegant, economical methods. Look for symmetry. Look for invariants.
- Sometimes symmetry and economy conflict. For example, it is not always best to square both sides of equations like $\sqrt{A}+\sqrt{B}=C$. Sometimes it makes better sense to change it to $\sqrt{A}=C-\sqrt{B}$, and then square. (Example: 2006B/15).
- Zero is your best friend. One is your second-best friend.
- Never multiply out, unless you have to. Always look for factorizations, instead.
- To know the zeros of a polynomial is to know the polynomial.
- Look for telescoping terms.
- Don't worry about being clever. Dumb methods work, too. Low-tech is better than high-tech. But if things start getting too dirty and messy, step back and ask yourself if there is a better way to proceed.

2 *Arithmetic*.

- Know all squares up to $\lceil\sqrt{2013}\rceil^2$
- Know all perfect powers < 2013
- Know all factorials up to 10!
- Know the first 9 or 10 rows of Pascal's Triangle
- Factor the current year!
- Know your primes, at least under 100, ideally up to 200 or so.
- Be able to mentally square numbers and multiply numbers, using the factorization $x^2-y^2=(x-y)(x+y)$. For example, to compute 73^2, use $73^2-3^2=70*76=4900+420=5320$.

3 *Factoring*.

- $1001=7\times 11\times 13$
- $(x+y)^2=x^2+2xy+y^2$.
- $(x-y)^2=x^2-2xy+y^2$.
- $(x+y)^3=x^3+3x^2y+3xy^2+y^3=x^3+y^3+3xy(x+y)$.
- $(x-y)^3=x^3-3x^2y+3xy^2-y^3=x^3-y^3-3xy(x-y)$.
- $x^2-y^2=(x-y)(x+y)$.
- $x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+y^{n-1})$ for all n.
- $x^n+y^n=(x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^2-\cdots+y^{n-1})$ for all odd n (the terms of the second factor alternate in sign).
- $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$

4 *Polynomials*.

- FTA, conjugate complex solutions
- Completing the square
- Factor/Remainder theorem
- Relationship between roots and coefficients

5 *Sequences and Series*.

- Arithmetic series: The best sum formula is $S = n(first + last)/2$, since it tends to get you thinking about averages which gets you thinking about symmetry.
- Know the formulas for sums of squares and cubes. Be able to derive formulas for sums of higher powers if needed.
- TELESCOPING is the mother of all sequence/summation methods. Know the classic telescopes, for example $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ and $(n+1)! - n! = n \cdot n!$. And don't forget that telescoping can be used with products as well. You can always turn products into sums with logarithms.
- You don't really need to learn the "calculus of differences" to handle recurrence sequence problems. Almost always, low-tech methods (usually involving telescoping!) suffice.
- Even though it is really a calculus topic, you should be able to prove that the harmonic series $1+1/2+1/3+1/4+\cdots$ diverges (without calculus). Likewise, you should be able to prove that $1+1/4+1/9+1/16+\cdots$ converges, again, without calculus.
- The sum of the first n odd numbers is equal to n^2. This is a remarkably fruitful fact.

6 *Miscellaneous*.

- Floor and ceiling functions
- Floor functions and counting multiples; lattice points.
- Absolute value
- Logarithms: Be able to prove the all-important, and easy-to-memorize formula $\log_a b \log_b c = \log_a c$.
- Complex numbers: Cis form, sums of roots of unity, take absolute value of both sides.

Mostly Algebra

1 *2004:7*. Let C be the coefficient of x^2 in the expansion of the product

$$(1-x)(1+2x)(1-3x)\cdots(1+14x)(1-15x).$$

Find $|C|$.

2 *2008 II:1*. Find the remainder when

$$100^2+99^2-98^2-97^2+96^2+\cdots+4^2+3^2-2^2-1^2$$

is divided by 1000.

3 For which integer n is $1/n$ closest to $\sqrt{1{,}000{,}000} - \sqrt{999{,}999}$?

4 *BAMM 2000*. Let $x = \sqrt[3]{1000} - \sqrt[3]{999}$. What integer is closest to $1/x$?

5 Solve $x^4+x^3+x^2+x+1=0$.

6 *2005 II: 7*. Let

$$x = \frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}.$$

Find $(x+1)^{48}$.

7 Factor z^5+z+1.

8 Solve $z^6+z^4+z^3+z^2+1=0$.

9 *2004:13*. The polynomial

$$P(x)=(1+x+x^2+\cdots+x^{17})^2-x^{17}$$

has 34 complex roots of the form $z_k=r_k[\cos(2\pi a_k)+i\sin(2\pi a_k)], k=1,2,3,\ldots,34$, with $0<a_1\le a_2\le a_3\le\cdots\le a_{34}<1$ and $r_k>0$. Given that $a_1+a_2+a_3+a_4+a_5=m/n$, where and are relatively prime positive integers, find $m+n$.

10 (USAMO 1975) If $P(x)$ denotes a polynomial of degree n such that $P(k)=k/(k+1)$ for $k=0,1,2,\ldots,n$, determine $P(n+1)$.

11 Find the minimum value of $xy+yz+xz$, given that x,y,z are real and $x^2+y^2+z^2=1$. No calculus, please!

12 Find all integer solutions (n,m) to

$$n^4+2n^3+2n^2+2n+1=m^2.$$

13 If $x^2+y^2+z^2=49$ and $x+y+z=x^3+y^3+z^3=7$, find xyz.

14 *2008 II:7*. Let r,s,t be the roots of $8x^3+1001x+2008=0$. Find $(r+s)^3+(s+t)^3+(t+r)^3$.

15 Find all real values of x that satisfy $(16x^2-9)^3+(9x^2-16)^3=(25x^2-25)^3$.

16 *2002 II:9*. Given that is a complex number such that $z+\frac{1}{z}=2\cos 3^\circ$, find the least integer that is greater than $z^{2000}+\frac{1}{z^{2000}}$.

17 (*Crux Mathematicorum*, June/July 1978) Show that n^4-20n^2+4 is composite when n is any integer.

18 *2008:4*. There exist unique positive integers x and y that satisfy the equation $x^2+84x+2008=y^2$. Find $x+y$.

19 *2005 II: 11*. Let m be a positive integer, and let $a_0,a_1,\ldots,a_m$ be a sequence of real numbers such that $a_0=37, a_1=72, a_m=0$, and

$$a_{k+1}=a_{k-1}-\frac{3}{a_k}$$

for $k=1,2,\ldots,m-1$. Find m.

20 Prove that

$$\sqrt{\frac{1}{\left(\frac{1}{1729}-\frac{22}{7}\right)^2}+\frac{1}{\left(\frac{22}{7}-\frac{355}{113}\right)^2}+\frac{1}{\left(\frac{355}{113}-\frac{1}{1729}\right)^2}}$$

is rational.

21 *2007I:14*. A sequence is defined as follows: $a_1=a_2=3$, and for $n\ge 2$ $a_{n+1}a_{n-1}=a_n^2+2007$. Find the greatest integer that does not exceed $\frac{a_{2006}^2+a_{2007}^2}{a_{2006}a_{2007}}$

22 (E. Johnston) Let S be the set of positive integers which do not have a zero in their base-10 representation; i.e.,

$$S=\{1,2,\ldots,9,11,12,\ldots,19,21,\ldots\}.$$

Does the sum of the reciprocals of the elements of S converge or diverge?

23 Let $A=\sum_{n=1}^{10000}\frac{1}{\sqrt{n}}$. Find $\lfloor A\rfloor$ without a calculator.

24 *2006II:15*. Given that x, y, z are real numbers that satisfy:

$$x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}$$
$$y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}$$
$$z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}}$$

and that $x + y + z = m/\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime, find $m + n$.

Number Theory Toolkit

1 Primes and Divisibility

FTA (Fundamental Theorem of *Arithmetic*) Every positive integer can be factored *uniquely* into primes. This factorization is often called the PPF (prime-power factorization).

GCD and LCM Greatest common divisor and least common multiple, respectively. If $\text{GCD}(a, b) = 1$ we say that a and b are **relatively prime** and denote this by $a \perp b$.

- If $g|a$ and $g|b$ then $g|(a+b), g|(a-b)$, etc. In fact, $g|(ax+by)$ for any integers x, y. The expression $ax + by$ is called a **linear combination** of a and b.
- Consequently, every pair of consecutive positive integers is relatively prime. Same is true of consecutive odd integers. Consecutive even integers have a GCD of 2.
- *Linear combination rule*: The GCD of a and b is a linear combination of a and b. In fact, it is smallest positive linear combination of a and b. Thus if $a \perp b$, there exist integers x, y such that $ax + by = 1$.
- The coefficients x, y above can be found by performing the *Euclidean algorithm* backwards. In practice, they can be found by inspection for small numbers. For example, $5 \perp 7$ and $3 \cdot 5 + (-2) \cdot 7 = 1$. (Don't worry if you don't know the Euclidean algorithm.)

Number of divisors is denoted by $\tau(n)$ and includes 1 and n in the count. So $\tau(p) = 2$ for any prime p and in general, $\tau(p^a q^b \cdots) = (a+1)(b+1)\cdots$ when the number is in PPF form. (Some books use $d(n)$ instead of $\tau(n)$ but Greek letters are so much more sophisticated.)

Sum of divisors is denoted by $\sigma(n)$ and includes 1 and n. The formula is rather simple. Make sure you can see why it's true (try simple examples such as $n = 12$ and $n = 300$):

$$\sigma(p^a q^b \cdots) = (1 + p + p^2 + \cdots + p^a)(1 + q + q^2 + \cdots + q^b)\cdots.$$

2 Modular Arithmetic

Congruence notation The notation $x \equiv y \pmod{m}$ means the following equivalent things. All of them are worth internalizing.

- $x - y$ is a multiple of m
- x has the remainder y when divided by m, provided that $0 \le y < m$.
- $x = mK + y$ for some integer K

Thus $x \equiv 3 \pmod 4$ means that x has a remainder of 3 when divided by 4, and it also means that you can write x "in the form" $4k + 3$. It is common to use small negative numbers in congruences, especially -1. For example, $99 \equiv -1 \pmod 4$. We could have written $99 \equiv 3 \pmod 4$, but the former is just as true, and often more useful.

Congruence algebra Congruence notation is incredibly useful because you can add, subtract, and multiply (*but not divide*) both just as you can with ordinary equality. For example,

- If $a \equiv b \pmod{m}$ and $x \equiv y \pmod{m}$, then $a + x \equiv b + y \pmod{m}$ and $ax \equiv by \pmod{m}$
- $100000 \equiv 10 \pmod{11}$, since $10 \equiv -1 \pmod{11}$ and thus $10^5 \equiv (-1)^5 \pmod{11}$.

Divisibility Rules You should know the divisibility rules for 2, 3, 4, 5, 6, 8, 9, and 11, using congruences or other methods.

Mod *n* analysis is a crucial start of any problem. Put it into mod 2, 3, and 4 filters, sometimes more. For example, you should verify that all perfect squares are congruent to 0 or 1 (mod 4) only. (A mod 2 analysis is also called a parity analysis.)

Fermat's Little Theorem states that if p is a prime, and a is not a multiple of p, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

For example, $4^{12} \equiv 1 \pmod{13}$.

3 Diophantine Equations

- Use of parity, mod 3, mod 4 analysis
- Linear equations
- $x^2 - y^2 = n$. The AIME uses this humble equation in endless ways. Become intimate with it.
- Pythagorean equation
- $x^2 + y^2 = n$

Mostly Number Theory

1 Show that if $a^2 + b^2 = c^2$, then $3|ab$.

2 If $x^3 + y^3 = z^3$, show that one of the three must be a multiple of 7.

3 Make sure that you know why 100! ends in 24 zeros and 1000! ends in 249 zeros. Can $n!$ end with $n/4$ zeros?

4 Find the smallest positive integer n such that $\tau(n) = 10$.

5 Find the remainder when 2^{1000} is divided by 13. (This was an AHMSE problem when I was in high school.)

6 *2006II:3*. Let P be the product of the first 100 positive odd integers. Find the largest integer k such that P is divisible by 3^k.

7 *BAMO 1999*. Prove that among any 12 consecutive positive integers there is at least one which is smaller than the sum of its proper divisors. (The proper divisors of a positive integer n are all positive integers other than 1 and n which divide n. For example, the proper divisors of 14 are 2 and 7.)

8 *BAMO 2000*. Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1. (Two integers are relatively prime if they share no common positive divisor other than 1. For example, 22 and 15 are relatively prime, and thus $37 = 22 + 15$ represents the number 37 in the desired way.)

9 What kind of numbers can be written as the sum of two or more consecutive integers? For example, 10 is such a number, because $10 = 1 + 2 + 3 + 4$. Likewise, $13 = 6 + 7$ also works.

10

11 *2000II:2*. A point whose coordinates are both integers is called a lattice point. How many lattice points lie on the hyperbola $x^2 - y^2 = 2000^2$?

12 *2000II:4*. What is the smallest positive integer with six positive odd integer divisors and twelve positive even divisors?

13 *BAMM 1999*. How many ordered pairs (x,y) of integers are solutions to

$$\frac{xy}{x+y} = 99?$$

14 *1991*. Find x^2+y^2 if $x,y \in \mathbf{N}$ and

$$xy+x+y = 71, \quad x^2y+xy^2 = 880.$$

15 *1985:13*. The numbers in the sequence

$$101, 104, 109, 116, \ldots$$

are of the form $a_n = 100+n^2$, where $n = 1,2,3,\ldots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

16 *2001II:10*. How many positive integer multiples of 1001 can be expressed in the form $10^j - 10^i$, where i, j are integers and $0 \le i < j \le 99$?

17 *2003II:10*. Two positive integers differ by 60. The sum of their square roots is the square root of an integer that is not a perfect square. What is the maximum possible sum of the two integers?

18 Find the last three digits of 7^{9999}.

19 Let $\{a_n\}_{n\ge 0}$ be a sequence of integers satisfying $a_{n+1} = 2a_n + 1$. Is there an a_0 so that the sequence consists entirely of prime numbers?

20 (USAMO 1979) Find all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ to

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1{,}599.$$

21 *2005:12*. For positive integers let $\tau(n)$ denote the number of positive integer divisors of n including 1 and n. Define $S(n) = \tau(1)+\tau(2)+\cdots+\tau(n)$. Let a denote the number of positive integers $n \le 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \le 2005$ with $S(n)$ even. Find $|a-b|$.

22 *2007II: 7*. For a certain integer k there are exactly 70 positive integers $n_1, n_2, \ldots, n_{70}$ such that

$$k = \lfloor \sqrt[3]{n_1} \rfloor = \lfloor \sqrt[3]{n_2} \rfloor = \cdots = \lfloor \sqrt[3]{n_{70}} \rfloor$$

and k divides n_i for all i such that $1 \le i \le 70$.

Find the maximum value of n_i/k for $1 \le i \le 70$.

23 *2004II:10*. Let S be the set of integers between 1 and 2^{40} whose binary expansions have exactly two 1's. If a number is chosen at random from S, the probability that it is divisible by 9 is m/n where m and n are relatively prime positive integers. Find $m+n$.

24 *2007II: 13*. A triangular array of squares has one square in the first row, two in the second, and in general, k squares in the kth row for $1 \le k \le 11$. With the exception of the bottom row, each square rests on two squares in the row immediately below (illustrated in diagram). In each square of the eleventh row, a 0 or a 1 is placed. Numbers are then placed into the other squares, with the entry for each square being the sum of the entries in the two squares below it. For how many initial distributions of 0's and 1's in the bottom row is the number in the top square a multiple of 3?

25 *2008II:15*. Find the largest integer satisfying the following conditions:

(i) n^2 can be expressed as the difference of two consecutive cubes; (ii) $2n+79$ is a perfect square.