Putnam Competition 2012, Session B

Evan O'Dorney

January 22, 2013

1. Let S be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

   (i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x+1)$ are in S;

   (ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

   (iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

   Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

2. Let P be a given (non-degenerate) polyhedron. Prove that there is a constant $c(P) > 0$ with the following property: If a collection of n balls whose volumes sum to V contains the entire surface of P, then $n > c(P)/V^2$.

3. A round-robin tournament among $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

4. Suppose that $a_0 = 1$ and that $a_{n+1} = a_n + e^{-a_n}$ for $n = 0, 1, 2, \ldots$. Does $a_n - \log n$ have a finite limit as $n \to \infty$? (Here $\log n = \log_e n = \ln n$.)

5. Prove that, for any two bounded functions $g_1, g_2 : \mathbb{R} \to [1, \infty)$, there exist functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$,

$$\sup_{s \in \mathbb{R}} (g_1(s)^x g_2(s)) = \max_{t \in \mathbb{R}} (xh_1(t) + h_2(t)).$$

6. Let p be an odd prime number such that $p \equiv 2 \pmod 3$. Define a permutation π of the residue classes modulo p by $\pi(x) \equiv x^3 \pmod p$. Show that π is an even permutation if and only if $p \equiv 3 \pmod 4$.