1 General remarks.

Below are some facts that may be helpful when solving olimpiad problems. Most are from the intro the a great book "The IMO Compendium" (one of its authors is BMC's Ivan Matic). Most topics mentioned are covered in more depth by a past BMC session. Simply googling the topic name + Berkeley Math Circle often produces a nice handout on the topic. The best way to get good at solving problems is to solve problems. We provide some, many more can be found by - you guessed it - googlong "Berkeley Math Circle practice problems".

2 Some useful facts.

2.1 Algebra.

(i) Obvious but good: $P(x) - P(y)$ is divisible by $x - y$.

The rational root theorem: If $P(x) = a_n x^n + \ldots + a_1 x_1 + a_0$ has a rational root $x = \frac{p}{q}$ (with p and q relatively prime) then p divides a_0 and q divides a_n.

(ii) Recurrence relations: If the sequence is x_n satisfies recurrnce relation $x_n + a_1 x_{n-1} + \ldots + a_k x_{n-k}$ then the associated polynomial is $P(x) = x^k + a_1 x^{k-1} + \ldots + a_k$. If $P(x)$ has roots $\alpha_1, \ldots, \alpha_k$ with multiplicity $k_1 \ldots k_n$ then the sequence itself is then given by $x_n = p_1(n)\alpha_1^n + p_1(n)\alpha_1^n + \ldots + p_k(n)\alpha_k^n$, where p_i are polynomials of degree at most $k_i - 1$. The coefficients of the polynomials are determined by initial conditions $x_1, \ldots, x_{k-1}$.

See BMC session "Linear recursive Sequences" by Bjorn Poonen.

Example: Fibonacci $x_n - x_{n-1} + x_{n-2} = 0$. $P(x) = x^2 - x - 1$ $\alpha_{1,2} = \frac{1 \pm \sqrt{5}}{2}$. Find coefficients to get formula for F_n.

Example 2: Recurrence $x_{n+2} = 4x_{n+1} - 4x_n$ with $x_0 = 0$ and $x_1 = 2$ gives $x_n = n2^n$; with $x_0 = 0$ and $x_1 = 1$ it gives $x_n = (n/2)2^n$ etc.

(iii) Inequalities: $QM > AM > GM > HM$.

Generalization: Let $M_p = \left(\frac{(x_1^p + \ldots + x_n^p)}{n}\right)^{1/p}$, with values for p equal $\pm\infty$ and 0 understood as limits. In particular $M_\infty = max(x_i)$, $M_2 = QM$, $M_1 = AM$, $M_0 = GM$, $M_{-1} = HM$ and $M_{-\infty} = min(x_i)$. Then M_p is increasing function of p (for fixed x_i).

(iv) Convex functions: $f : I \mapsto \mathbb{R}$ is convex if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$. If f is continuous this is equivalent to $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$.*

Jensen inequality: For a convex function $f(a_1 x_1 + \ldots a_k x_k) \leq a_1 f(x_1) + \ldots a_k f(x_k)$ for all non-negative a_i summing to 1.

(v) For a tuple $a = (a_1, \ldots a_k)$ of positive real (but in practice mostly integer) numbers let $T_a(x_1, \ldots, x_n) = \sum y_1^{a_1} \ldots y_k^{a_k}$ where the sum is over all permutations y_i of the x_i. We say that a majorizes b if the "diagram for b can be obtained by falling stuff from diagram for a".† Then $T_a(x_1, \ldots x_n) \geq T_b(x_1, \ldots x_n)$.

(vi) See "Some notes on Inequalities" BMC session by Ngoc Mai Tran.

2.2 Number theory.

(i) Modular arithmetic/infinite descent: a) A solution of a Diophantine equation must also be a solution mod p for every prime p b) Assume the solution is minimal for a contradiction.

Example: Fermat theorem for $n = 4$. Show $x^4 + y^4 = z^4$ has no positive solutions.

* If f is twice differentiable this is $f'' \geq 0$.

†Formally, if $a_1 + \ldots a_n = b_1 + \ldots b_n$ and $a_1 + \ldots a_k \geq b_1 + \ldots b_k$ for each $k = 1, \ldots n-1$.

(ii) Extended Euclid algorithm: For fixed integer a, b and k the equation $ma + lb = k$ has integer solutions m, l if and only if $GCD(a, b)$ divides k. For $k = GCD(a, b)$ a solution may be found by using extended Euclid's algorithm.

(iii) Chinese Reminder Theorem: If $d_1, \ldots d_k$ are pairwise relatively prime, $GCD(a_i, d_i) = 1$ then the system of congruences $a_i x \equiv c_i \pmod{d_i}$ has unique solution modulo $m_1 m_2 \ldots m_k$.

(iv) Fermat's little theorem: For a prime p we have $a^{p-1} \equiv 1 \pmod p$ for all a not divisible by p.

Euler's theorem: If a is relatively prime to n then $a^{\phi(n)} \equiv 1 \pmod n$. Here $\phi(n)$ is Euler's totient function.

Wilson's theorem: For prime p we have $(p-1)! \equiv -1 \pmod p$.

Advanced: Pell's equation (see BMC session "Pell's Equation" by Ian Le), quadratic reciprocity.

2.3 Combinatorics.

(i) Pigeonhole principle. Permutations. Combinations.

Example: How many ways of partitioning n into k non-negative summands are there? Into positive summands?

(ii) Inclusion-exclusion principle.

See for example the BMC session "Combinatorics and Geometry" session by Vera Serganova.

(iii) Graph theory: Number of odd degree vertices in any graph is even. A tree of n vertices has $n - 1$ edges.

Euler path - a path visiting each edge exactly once - exists if and only if the number of odd degree vertices is 2 or 0. If it's 0 then such a path is closed.

Euler formula: For a planar graph $E + 2 = F + V$ where F is the number of faces including the "outer infinite face." Corollary: A planar graph has a vertex of degree at most 5.

See BMC session "Graph Theory and Ramsey Theory" by Paul Zeitz.

(iv) Invariants and monovariants (see Gabriel Carrol's "Monovariants" BMC session).

(v) Colorings. See for example Tom Davis's session on coloring.

2.4 Geometry.

(i) "Angle chase" : Opposite angles of inscribed quadrilateral add up to 180. If two inscribed angles intercept the same arc, then the angles are equal. This is probably the most used stuff in geometry problems.

(ii) Vectors (see BMC session on "Vectors -Applications to Problem Solving" Zvezdelina Stankova).

(iii) Ceva heorem: In a triangle ABC if A_1 is on the segment BC, B_1 on CA and C_1 on AB then the segments AA_1, BB_1 and CC_1 intersect at one point if and only if $\frac{AC_1}{C_1B}\frac{BA_1}{A_1C}\frac{CB_1}{B_1A} = 1$.

Menelaus theorem: As before but now C_1 is on the continuation of AB and the points A_1, B_a and C_1 are collinear if and only if $\frac{AC_1}{C_1B}\frac{BA_1}{A_1C}\frac{CB_1}{B_1A} = -1$.

These are best proved using masses (see Tom Rike's BMC session on "Mass Point Geometry") or barycentric coordinates (see Evan Chen's BMC session "Barycentric Coordinates"), both of which are important on their own.

(iv) Power of a point: The degree of point A with respect to a circle centered at point O or radius R is defined as $OA^2 - R^2$. The radical axis of two circles is the set of all points which have the same power with respect to the two circles. The theorem is that this is a line (see Zvezdelina Stankova's BMC session"Inversion in the Plane".)

(v) Ptolemy theorem: It $ABCD$ is an inscribed quadrilateral then $AC \cdot BD = AB \cdot CD + BC \cdot DA$.

Ptolemy inequality: For any 4 points $AC \cdot BD \leq AB \cdot CD + BC \cdot DA$

(vi) Law of sines: In the triangle ABC one has $\frac{BC}{\sin A} = \frac{CA}{\sin B} = \frac{AB}{\sin C} = 2R$.

(vii) Advanced geometry:

Inversion (see Zvezdelina Stankova's BMC session "Inversion").

Euler's line, 9 point circle, Euler's formula: $OI^2 = R(R - 2r)$

3 Some problems.

1 (USAMO 1974) Does there exist a polynomial P with integer coefficients and distinct integers a, b, c such that $P(a) = b, P(b) = c$ and $P(c) = a$.

2 (BAMO 2004) Find (with proof) all monic polynomials $f(x)$ with integer coefficients that satisfy the following two conditions.

(a) $f(0) = 2004$.

(b) If x is irrational, then $f(x)$ is also irrational.

3 (USAMO 1974) For $a, b > 0$ prove $a^a b^b c^c \geq (abc)^{(a+b+c)/3}$.

4 Prove $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq 3/2$.

5 Prove $2\sqrt{a} + 3\sqrt[3]{b} > 5\sqrt[5]{ab}$.

6 (USAMO 1976) Find all integer solutions of $a^2 + b^2 + c^2 = a^2 b^2$.

7 Divide 19 degree angle into 19 equal parts with straight-edge and compass.

8 Find n consecutive integers that are divisible by squares.

9 A circle is divided by n points into equal arcs.What is the number of closed polygonal chains with equal sides and vertices in these points?

10 (BAMO 2010) Place eight rooks on a standard $8x8$ chessboard so that no two are in the same row or column. With the standard rules of chess, this means that no two rooks are attacking each other. Now paint 27 of the remaining squares (not currently occupied by rooks) red. Prove that no matter how the rooks are arranged and which set of 27 squares are painted, it is always possible to move some or all of the rooks so that:

- All the rooks are still on unpainted squares.
- The rooks are still not attacking each other (no two are in the same row or same column).
- At least one formerly empty square now has a rook on it. That is, the rooks are not on the same 8 squares as before.

11 (BAMO 2005) There are 1000 cities in the country of Euleria, and some pairs of cities are linked by dirt roads. It is possible to get from any city to any other city by traveling along these roads. Prove that the government of Euleria may pave some of the roads so that every city will have an odd number of paved roads leading out of it.

12 Initially, there is one pile with 100 coins in it. A player plays the following game: In each step she chooses one of the piles that have more than one coins, splits the pile into two, and writes on the blackboard the product of the number of coins in the smaller piles. The game ends when all the remaining piles have one coin each. In the end of the game, the sum of all the numbers written on the blackboard is calculated. What is the maximal sum the player can obtain?

13 (BAMO 2000) Alice plays the following game of solitaire on a 20 x 20 chessboard. She begins by placing 100 pennies, 100 nickels, 100 dimes, and 100 quarters on the board so that each of the 400 squares contains exactly one coin. She then chooses 59 of these coins and removes them from the board. After that, she removes coins, one at a time, subject to the following rules:

- A penny may be removed only if there are four squares of the board adjacent to its square (up, down, left, and right) that are vacant (do not contain coins). Squares "off the board" do not count towards this four: for example, a non-corner square bordering the edge of the board has three adjacent squares, so a penny in such a square cannot be removed under this rule, even if all three adjacent squares are vacant.
- A nickel may be removed only if there are at least three vacant squares adjacent to its square. (And again, "off the board" squares do not count.)
- A dime may be removed only if there are at least two vacant squares adjacent to its square ("off the board" squares do not count).
- A quarter may be removed only if there is at least one vacant square adjacent to its square ("off the board" squares do not count).

Alice wins if she eventually succeeds in removing all the coins. Prove that it is impossible for her to win.

14 (BAMO 2008) Determine the greatest number of z- tetrominoes that can be placed in a 9×9 grid (without overlapping), such that each tetromino covers exactly 4 unit squares.

15 All the chairs in a classroom are arranged in a square $n \times n$ array (in other words, n columns and n rows), and every chair is occupied by a student. The teacher decides to rearrange the students according to the following two rules:

(a) Every student must move to a new chair.

(b) A student can only move to an adjacent chair in the same row or to an adjacent chair in the same column. In other words, each student can move only one chair horizontally or vertically.

(Note that the rules above allow two students in adjacent chairs to exchange places.) Show that this procedure can be done if n is even, and cannot be done if n is odd.

16 Let D be a point on the side BC of triangle ABC, let DF and DE be the perpendiculars to BA and AC respectively. Where on BC should D be positioned to minimize the length of EF?

17 (BAMO 2008) Point D lies inside the triangle ABC. If A_1, B_1, and C_1 are the second intersection points of the lines AD, BD, and CD with the circles circumscribed about BDC, CDA, and ADB, prove that $\frac{AD}{AA_1} + \frac{BD}{BB_1} + \frac{CD}{DD_1} = 1$.