Berkeley Math Circle
Monthly Contest 8
Due May 7, 2013

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 7
by Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Two frogs are sitting at opposite ends of a row of 10 lily pads. Every second, they simultaneously hop to an adjacent lily pad. Is it possible for them to come to rest simultaneously on the same lily pad?

2. Let
$$x = \sqrt[3]{2+\sqrt{3}} + \sqrt[3]{2-\sqrt{3}}.$$
Prove that $x^3 = 3x + 4$.

3. Do there exist nonnegative integers $a_1, a_2, \ldots, a_{10}$ such that the positive integer
$$1^{a_1} \cdot 2^{a_2} \cdot 3^{a_3} \cdot 4^{a_4} \cdot 5^{a_5} \cdot 6^{a_6} \cdot 7^{a_7} \cdot 8^{a_8} \cdot 9^{a_9} \cdot 10^{a_{10}}$$
ends with the digits 11?

4. Let ABC be a triangle, and let M be the midpoint of BC. Prove that
$$AM < \frac{AB + AC}{2}.$$

5. Find all positive integers that are the difference of two triangular numbers in exactly one way. (A *triangular number* is one of the numbers $0, 1, 3, 6, \ldots$ having the form $(n^2 + n)/2$, $n \geq 0$.)

6. Determine, with proof, the largest positive integer k such that 2^k divides $2013^{2^{2013}} - 1$.

7. Jimmy is jumping on a pogo stick on a flat surface delimited on one side by a barn (which may be idealized as a straight line). Each jump he makes has length 1 foot and brings him closer to the barn. Prove that after 2013 jumps, Jimmy is at least 1 foot away from where he started.