Berkeley Math Circle
Monthly Contest 6 – Solutions

1. A $10 \times 10 \times 10$-inch wooden cube is painted red on the outside and then cut into its constituent 1-inch cubes. How many of these small cubes have at least one red face?

   Solution 1. There are 8 corner cubelets, each with three red faces. Each of the twelve edges has a row of 8 edge cubelets, each with two red faces. The cubelets with only one red face form 8×8 squares in the interior of each of the six faces. So the total count is
   $$8 + 12 \cdot 8 + 6 \cdot 8 \cdot 8 = 8 + 96 + 384 = 488.$$

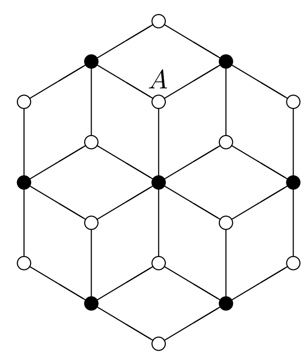
   Solution 2. Let us focus on the cubelets that do *not* have a red face. To remove all the cubelets whose top face is red, we can cut a 1-inch slab off of the top of the cube. Similarly, we can remove 1-inch slabs from the other five faces of the cube to get rid of the cubelets with red on those faces. After this procedure, each of the cube's three dimensions has been reduced by 2; we thus have a cube of $8 \cdot 8 \cdot 8 = 512$ cubelets with no paint. To count the cubelets that *do* have red paint, then, we subtract from the total of $10 \cdot 10 \cdot 10 = 1000$ to arrive at 488 cubelets.

2. The whole numbers from 1 to 100 are each written on an index card, and the 100 cards shuffled in a hat. Twenty-six cards are drawn out of the hat at random. Prove that two of the numbers drawn have a difference of 1, 2, or 3.

   Solution 1. Divide the numbers from 1 to 100 into groups of four:
   $$(1 \quad 2 \quad 3 \quad 4) \quad (5 \quad 6 \quad 7 \quad 8) \quad \cdots \quad (97 \quad 98 \quad 99 \quad 100)$$
   Since $100/4 = 25$, there are 25 groups. If 26 numbers are selected, then two of them must come from the same group (the famous "Pigeonhole Principle"), and so their difference is at most 3.

   Solution 2. Assume to the contrary that every pair of cards drawn has a difference of at least 4. Arrange the 26 cards from least to greatest. The first card is, of course, greater than or equal to 1. The second card is greater than or equal to $1 + 4 = 5$, and the third card is at least $5 + 4 = 9$. Continuing this way, we find that the 26th card is greater than or equal to
   $$1 + 25 \cdot 4 = 101,$$
   which is a contradiction.

3. A science museum has the design shown, with 19 exhibits (shown as dots) connected by 30 hallways.

   (a) Is there a route that travels along each hallway exactly once? (Some exhibits may be visited multiple times.)

   (b) Is there a route that visits each exhibit exactly once? (Some hallways may remain unused.)

   Solution. (a) The answer is no. Consider an exhibit connected to three hallways, such as the one marked A above. Assume that A is not the first exhibit on the path. Then at some point we enter A along some hallway and exit it by another hallway. There is still one more hallway to visit, so at some later time we enter A by it and—as there are no more hallways at A—the path ends.

   Thus A must be either the first or the last exhibit on the desired path. But the same can be said of each of the six exhibits with three hallways, so no such path exists.

   (b) The answer is again no. Color the exhibits black and white as shown in the diagram. Note that every edge connects a black and a white node. Therefore, on such a path, the black and white nodes must alternate. Since there are 7 black and 12 white exhibits, this is impossible.

4. Five consecutive vertices of a regular 2013-gon are given. Prove that one can reconstruct the entire 2013-gon using straightedge alone.

   Solution. Let A, B, C, D, E, and F be six consecutive vertices of the polygon. We prove that, given A, B, C, D, and E, it is possible to construct F with straightedge alone. Then, continuing around the polygon, we can construct all the vertices and then fill in the sides.

   Our proof is based on the observation that the polygon has a line ℓ of symmetry such that the pairs A and F, B and E, C and D are reflections with respect to that line.

   The construction proceeds as follows:

   - Let BC and DE meet at X; let BD and CE meet at Y; join XY. This is the line ℓ of symmetry.
   - Let AB meet XY at U; join UE (this will pass through F, as AB and EF are symmetric about ℓ);
   - Let AC meet XY at V; join VD (this likewise passes through F).
   - The lines UE and VD meet at the point F we seek. (They do not coincide since D, E, F are noncollinear.)

   Remark. Using advanced techniques ("all quadrilaterals are projectively equivalent"), it is possible to prove that four points A, B, C, D are not enough to reconstruct the polygon.

5. Determine whether there is a polynomial $f(x)$ such that

   - Every coefficient of f, from the leading coefficient down to the constant term, is either 1 or -1.
   - $(x-1)^{2013}$ evenly divides $f(x)$ (this means that their quotient is a polynomial).

   Solution. The answer is yes. We replace 2013 by a variable n and prove that such a polynomial exists for all n. Our base case is $n = 0$, for which the polynomial $f_0(x) = 1$ clearly works.

   Given a polynomial $f_n(x)$ of degree d that is divisible by $(x-1)^n$, consider the polynomial

   $$f_{n+1}(x) = f_n(x)(x^{d+1} - 1).$$

   When the multiplication is expanded, we get one term of each of the orders $x^{2d+1}, x^{2d}, \ldots, x, 1$, each appearing with either a positive or a negative sign. This establishes condition (a). As for condition (b), we see that $x^{d+1} - 1$ is divisible by $x - 1$ (since $x = 1$ is a root of it) and $f_n(x)$ is divisible by $(x-1)^n$, hence $f_{n+1}(x)$ is divisible by $(x-1)^{n+1}$.

6. Let $a_1, a_2, \ldots, a_{2013}$ be real numbers satisfying the following conditions:

   - $a_1 = a_{2013} = 0$;
   - $|a_i - a_{i+1}| < 1$, for $1 \le i \le 2012$.
   - $\sum_{i=1}^{2013} a_i = 0$.

   Find the greatest possible value of the sum $\sum_{i=1}^{m} a_i$, where m ($1 \le m \le 2013$) is allowed to vary, in addition to the sequence $\{a_i\}$.

   Solution. The answer is that the sum can come arbitrarily close to $503^2 = 253009$, but (due to an unfortunate oversight in the problem statement) cannot equal it.

   Suppose that the sequence $\{a_i\}$ is fixed, and consider the m that yields the largest value of the sum $\sum_{i=1}^{m} a_i$. Clearly, we can assume $1 \le m \le 2012$. If $m \ge 1007$, consider the alternative sequence $\{a'_i\}$ defined by

$$a'_i = -a_{2014-i}.$$

   The partial sums of this sequence and the original one are related by

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{2013-m} a'_i,$$

   so by passing from one sequence to the other we can assume that the greatest sum occurs for $m \le 1006$.

   The conditions obviously imply

$$a_2 < 1, a_3 < 2, \ldots, a_i < a_{i-1}. \tag{1}$$

   On the other hand, note that $a_{m+1} \le 0$, since otherwise a better value of m would be $m+1$. So

$$a_m < 1, a_{m-1} < 2, \ldots, a_i < a_{m+1-i}. \tag{2}$$

   If $m \le 504$, we use (1) to write $a_1 + \cdots + a_m < 0 + 1 + \cdots + 503$. Otherwise, we use (1) on $a_1, \ldots, a_{504}$ and (2) on the remaining terms (one can make the cut at $1 + \lfloor m/2 \rfloor$ instead, but the expressions become needlessly uglier) to write

$$\begin{aligned} a_1 + \cdots + a_m &< 0 + 1 + \cdots + 502 + 503 + (m - 504) + (m - 505) + \cdots + 1 \\ &\le 0 + 1 + \cdots + 502 + 503 + 502 + \cdots + 1 \\ &= \frac{503 \cdot 504}{2} + \frac{502 \cdot 503}{2} = 503^2. \end{aligned}$$

   One can indeed achieve this bound to arbitrary precision by taking

$$\begin{gathered} a_1 = 0, a_2 = x, a_3 = 2x, \ldots, a_{503} = 502x, a_{504} = 503x, a_{505} = 502x, \ldots, a_{1006} = x, a_{1007} = 0, \\ a_{1008} = -x, \ldots, a_{1509} = -502x, a_{1510} = -503x, a_{1511} = -502x, \ldots, a_{2012} = -x, a_{2013} = 0 \end{gathered}$$

   where x is a real number that may be brought arbitrarily close to 1.

7. Find all positive integers n such that $n \mid 2^n - 1$.

   Solution. The only such n is $n = 1$. Clearly $1 \mid 2^1 - 1$.

   Suppose that $n > 1$ is a solution. Let p be the smallest prime divisor of n. Note that $p > 2$ (an even value of n cannot divide $2^n - 1$). By Fermat's little theorem, $2^{p-1} \equiv 1 \pmod{p}$. But we are also given $2^n \equiv 1 \pmod{p}$. Thus the order m of 2 mod p (that is, the smallest m such that $2^m \equiv 1 \pmod{p}$) divides both $p - 1$ and n. But $p - 1$ has only prime factors less than p, and n has only prime factors greater than or equal to p, so these numbers are relatively prime. We get $m = 1$, so $2^1 \equiv 1 \pmod{p}$, a contradiction.