Berkeley Math Circle
Monthly Contest 4 – Solutions

1. Prove that every power of 3, from 27 onward, has an even tens digit.

   Solution. By repeatedly multiplying by 3, we see that the units digits of powers of 3 are either 3, 9, 7, or 1. Suppose that N is a power of 3 with an even tens digit; we will prove that the tens digit of $3N$ is also even, from which it will follow inductively that every power of 3 from 27 onward has an even tens digit.

   If N ends in 1 or 3, then when N is tripled, there will be no carrying from the units place to the tens place. Then the tens digit of $3N$ will arise from tripling the tens digit of N and hence will be even.

   If N ends in 7 or 9, then there will be a carry of 2 from the units place to the tens place. The tens digit of $3N$ will arise from tripling the tens digit of N and hence will still be even.

2. A positive integer is called *oddly even* if the sum of its digits is even. Find the sum of the first 2013 oddly even numbers.

   Solution. For convenience, we declare 0 to be oddly even (its digit sum is of course 0), and we sum the first 2014 oddly even nonnegative integers.

   Let us look at the oddly even numbers in a given hundred, that is, in the range $100n$ to $100n + 99$ where $n \geq 0$ is an integer. For each tens digit, there are five possibilities for the units digit, and for each units digit, there are five possibilities for the tens digit. In particular, there are exactly 50 oddly even numbers in this range. When we sum their units digits, we get

   $$5 \cdot 0 + 5 \cdot 1 + \cdots + 5 \cdot 9$$

   which is exactly half the sum of the units digits of all integers in the range $[100n, 100n + 99]$. The same can be said for the tens digits and all preceding digits, so the sum of all oddly even numbers in the range $[100n, 100n + 99]$ is half the sum of all the integers in this range.

   We deduce that the interval $[0, 3999]$ has 2000 oddly even numbers and their sum is

   $$\frac{1}{2}(0 + 1 + \cdots + 3999) = \frac{1}{2} \cdot \frac{3999 \cdot 4000}{2} = 3999 \cdot 1000 = 3999000.$$

   We compute the next fourteen oddly even numbers:

   $$4000, 4002, 4004, 4006, 4008, 4011, 4013, 4015, 4017, 4019, 4020, 4022, 4024, 4026.$$

   Their sum can be computed by hand using various expendients, for instance:

   $$\begin{aligned} & 14 \cdot 4000 + (0 + 2 + 4 + 6 + 8 + 11 + 13 + 15 + 17 + 19 + 20 + 22 + 24 + 26) \\ = {} & 14 \cdot 4000 + [(2 + 8) + (4 + 6) + (11 + 19) + (13 + 17) + (24 + 26) + 20 + 15 + 22] \\ = {} & 14 \cdot 4000 + [10 + 10 + 30 + 30 + 20 + 50 + 37] = 56000 + 187 = 56187. \end{aligned}$$

   So the total sum of the first 2013 oddly even numbers is

   $$3999000 + 56187 = 4055187.$$

3. The expression

$$(1 \quad 1 \quad 1 \quad \cdots \quad 1)$$

is written on a board, with 2013 ones in between the outer parentheses. Between each pair of consecutive ones you may write either "$+$" or "$)($" (you cannot leave the space blank). What is the maximum possible value of the resulting expression?

Solution. The answer is 3^{671}, formed by dividing the ones into groups of three as the answer suggests.

It should be obvious that a maximum value *exists*, since there are only finitely many ways to place the pluses and parentheses and one (or more) of them must give the largest value.

If there is a factor greater than or equal to 5, we can prove that the arrangement is *not* optimal by splitting off two 1's, e.g. changing

$$(1+1+1+1+1) \quad \text{to}(1+1)(1+1+1).$$

This is tantamount to changing a factor of $n+2$ to $2n$ for $n \geq 3$, which increases the value of the expression.

If we have a factor containing only *one* 1, then we can merge it with any other factor n; this changes n to $n+1$ and therefore also increases the value of the expression.

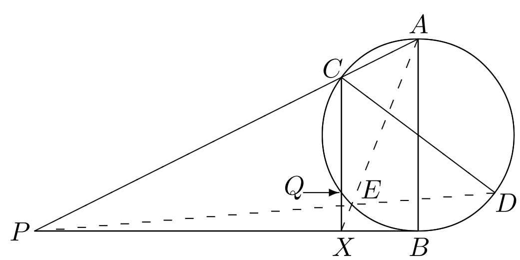
Therefore we need only consider expressions in which every factor is 2, 3, or 4. Because 4 can be broken up into $2 \cdot 2$ without changing the value of the expression, we will restrict our attention to expressions made up of 2's and 3's.

Now if there are at least three factors of 2, we can change them to two factors of 3 without altering the total number of 1's in the expression. This replaces a factor of 8 by a factor of 9, thus increasing the expression.

So the optimal way of forming the expression consists of zero, one, or two 2's and some undetermined number of 3's. Given that the total number of ones is 2013, it is clear that there must be no twos and 671 threes, implying that 3^{671} is the maximum value.

4. Let AB and CD be two nonperpendicular diameters of a circle centered at O, and let Q be the reflection of D about AB. The tangent at B meets AC at P, and DP meets the circle again at E. Prove that lines AE, BP, and CQ are concurrent.

Solution.



Let X be the intersection of CQ and BP. We first note that $CQ \parallel AB$ since

$$\angle CQA = \angle CBA = \angle BAD = \angle BAQ.$$

Note that $\triangle CPX \sim \triangle CDA$ since the angles at X and A are right and

$$\angle CPX = 90 - \angle CAB = \angle CBA = \angle BDA.$$

So $\triangle CPX \sim \triangle CDA$; rearranging the known facts

$$\frac{CP}{CX} = \frac{CD}{CA} \quad \text{and} \quad \angle PCX = \angle DCA$$

yields $\triangle CPD \sim \triangle CXA$. In particular, $\angle CAX = \angle CDP = \angle CAE$, so A, E, X are collinear, as desired.

5. Let $P(x)$ be a polynomial such that for all integers $x \geq 1$,

$$P(x) = \sum_{n=1}^{x} n^{2012}.$$

(a) Find $P(-2)$.

(b) Find $P(1/2)$.

Solution. (a) Consider the relation

$$P(x) = P(x-1) + x^{2012}. \tag{1}$$

If x is an integer greater than 1, then (1) clearly holds. Therefore (1) holds for all real x (two polynomials cannot have infinitely many common values unless they are the same). We can therefore use the recurrence $P(x-1) = P(x) - x^{2012}$ to compute

$$\begin{aligned} P(0) &= P(1) - 1^{2012} = 1 - 1 = 0 \\ P(-1) &= P(0) - 0^{2012} = 0 - 0 = 0 \\ P(-2) &= P(-1) - (-1)^{2012} = 0 - 1 = -1. \end{aligned}$$

(b) Consider the relation

$$P(-1-x) = -P(x). \tag{2}$$

The $x = 0$ case of this follows easily from the computations performed in part (a). Given this, we can easily prove (2) by induction for all nonnegative integers x using (1):

$$\begin{aligned} P(-2-x) &= P(-1-x) - (-1-x)^{2012} \\ &= -P(x) - (x+1)^{2012} \\ &= -P(x+1). \end{aligned}$$

Plugging $x = -1/2$ into (2) yields $P(-1/2) = 0$, so by (1),

$$P(1/2) = P(-1/2) + \left(\frac{1}{2}\right)^{2012} = \frac{1}{2^{2012}}.$$

6. How many functions $f \colon \mathbb{Z} \to \mathbb{R}$ satisfy the following three properties?

(a) $f(1) = 1$;

(b) For all $m, n \in \mathbb{Z}$, $f(m)^2 - f(n)^2 = f(m+n)f(m-n)$;

(c) For all $n \in \mathbb{Z}$, $f(n) = f(n + 2013)$.

Solution. By plugging $m = n = 0$ into (b) we easily get $f(0) = 0$. For any $u \in \mathbb{Z}$, we have

$$\begin{aligned} f(u+1)^2 - f(u-1)^2 &= f(2u)f(2) \\ f(u+1)^2 - f(u)^2 &= f(2u+1)f(1) = f(2u+1) \\ f(u)^2 - f(u-1)^2 &= f(2u-1)f(1) = f(2u-1) \end{aligned}$$

whence

$$f(2u)f(2) = f(2u+1) + f(2u-1).$$

We would like to conclude that

$$f(n+1) + f(n-1) = f(2)f(n)$$

for all $n \in \mathbb{Z}$. This is indubitable if n is even; otherwise we may use (c) and the fact that $n + 2013$ is even.

For any given value of $t = f(2)$, there is a unique function f satisfying the recursive definition

$$f(1) = 1, \quad f(2) = t, \quad f(n+1) + f(n-1) = tf(n). \tag{3}$$

If $t \neq \pm 2$, this solution is given by

$$f(n) = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \quad \text{where } \lambda_{1,2} \in \mathbb{C} \text{ are the roots of } \lambda^2 - t\lambda + 1 = 0. \tag{4}$$

Those familiar with the theory of linear recurrences will know a heuristic derivation of this formula. For our purposes it suffices to note that this function f does indeed satisfy definition (3) and in fact the condition (b) as well; thus the problem is to find out how many values of t cause condition (c) to hold.

If $t = \pm 2$, the solution (4) is invalid due to the fact that $\lambda_1 = \lambda_2$. In these cases the corresponding functions f satisfying (3) are $f(n) = n$ and $f(n) = (-1)^{n+1}n$, both of which fail condition (c) and hence can be discarded.

From the condition $f(2013) = f(0) = 0$, we derive that $\lambda_1^{2013} = \lambda_2^{2013} = \lambda_1^{-2013}$, so $\lambda_1^{4026} = 1$. We must have $\lambda^{2013} = 1$ or $\lambda^{2013} = -1$. If the latter holds, then from $f(2014) = f(1)$ we get

$$\begin{aligned}
\lambda_1^{2014} - \lambda_2^{2014} &= \lambda_1 - \lambda_2 \\
\lambda_1 \cdot \lambda_1^{2013} - \lambda_2 \cdot \lambda_2^{2013} &= \lambda_1 - \lambda_2 \\
-\lambda_1 + \lambda_2 &= \lambda_1 - \lambda_2 \\
\lambda_1 &= \lambda_2,
\end{aligned}$$

a contradiction. So λ_1, and hence its reciprocal λ_2, are 2013th roots of unity, a condition that is clearly sufficient to imply (c).

The trivial root $\lambda_1 = \lambda_2 = 1$ must be discarded. The remaining roots come in 1006 conjugate pairs yielding 1006 distinct real values of t. We conclude that there are 1006 such functions f.

7. Find all composite positive integers n such that all the divisors of n can be written in the form $a^r + 1$, where a and r are integers with $a \geq 0$ and $r \geq 2$.

Solution. The only such number is $n = 10$. It is easy to see that $n = 10$ indeed satisfies the conditions. Call n "good" if every divisor of n has the form $a^r + 1$, $a \geq 0$, $r \geq 2$ (a good n may be prime or composite).

First, it is easy to check that 4 is not good, and so 4 does not divide any good number.

Second, we show that a good integer is one more than a perfect square. Write $n = a^r + 1$ with r maximal. If r is odd, $a + 1$ divides n, and so $a + 1 = a_1^{r_1} + 1$. Hence $a = a_1^{r_1}$ and $n = a_1^{rr_1+1}$, violating the maximality assumption of r. Thus r must be even. We conclude that $n = x^2 + 1$ for some positive integer x.

Third, we show that the product of two odd primes are not good. Assume on the contrary that pq is good, where $p \leq q$ are odd primes. Then both p and q are good. We write $p = s^2+1$, $q = t^2+1$, and $pq = u^2+1$ (so $s \leq t < u$). Then $s^2t^2+s^2+t^2 = u^2$ or

$$s^2 q = s^2(t^2 + 1) = (u - t)(u + t) : \tag{5}$$

Since $q > s^2$ and q is prime, q must divide $u + t$. Note that both t and u are odd, so $u + t$ is even. Hence $2q = 2(t^2 + 1)$ divides $u + t$. It follows that $u + t \geq 2(t^2 + 1) = 2q$, and so $u - t \geq 2t^2 + 2 - 2t = t^2 + 1 + (t - 1)^2 > s^2$, contradicting (5). Hence our assumption was wrong, and so no good number is divisible by a product of two odd primes.

It follows that any n satisfying the conditions of the problem is of the form $n = 2p$, where p is an odd prime. We can write $n = x^2 + 1$ and $p = y^2 + 1$. Hence $x^2 + 1 = 2y^2 + 2$ or $p = y^2 + 1 = x^2 - y^2 = (x - y)(x + y)$. This is true only if $x = y + 1$, implying that $y = 2$, $p = 5$, and $n = 10$.