Berkeley Math Circle
Monthly Contest 4
Due January 8, 2013

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 4
by Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Prove that every power of 3, from 27 onward, has an even tens' digit.

2. A positive integer is called *oddly even* if the sum of its digits is even. Find the sum of the first 2013 oddly even numbers.

3. The expression
$$(1 \quad 1 \quad 1 \quad \cdots \quad 1)$$
is written on a board, with 2013 ones in between the outer parentheses. Between each pair of consecutive ones you may write either "+" or ")(" (you cannot leave the space blank). What is the maximum possible value of the resulting expression?

4. Let AB and CD be two nonperpendicular diameters of a circle centered at O, and let Q be the reflection of D about AB. The tangent at B meets AC at P, and DP meets the circle again at E. Prove that lines AE, BP, and CQ are concurrent.

5. Let $P(x)$ be a polynomial such that for all integers $x \geq 1$,
$$P(x) = \sum_{n=1}^{x} n^{2012}.$$
   (a) Find $P(-2)$.
   (b) Find $P(1/2)$.

   Remark. It is possible to prove that such a polynomial P exists and is unique; but this is not required for the problem.

6. How many functions $f\colon \mathbb{Z} \to \mathbb{R}$ satisfy the following three properties?

   (a) $f(1) = 1$;

   (b) For all $m, n \in \mathbb{Z}$, $f(m)^2 - f(n)^2 = f(m+n)f(m-n)$;

   (c) For all $n \in mathbbZ$, $f(n) = f(n + 2013)$.

7. Find all composite positive integers n such that all the divisors of n can be written in the form $a^r + 1$, where a and r are integers with $a \geq 0$ and $r \geq 2$.