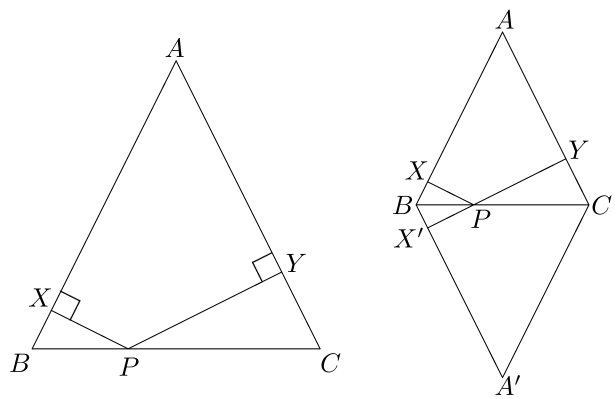
Berkeley Math Circle
Monthly Contest 2 – Solutions

1. There are 25 people at a party and every pair of them is either friends or strangers. Prove that there are two people at the party who have the same number of friends.

   Solution. Assume that no two people have the same number of friends. Because the number of friends a person can have ranges from 0 to 24, we can label the people with their numbers of friends and every label will be used once. Now consider the people labeled 0 and 24. They cannot be friends, because the person labeled 0 has no friends; but they cannot be strangers, because the person labeled 24 is friends with everybody else. We have a contradiction.

2. Let ABC be an isosceles triangle with $AB = AC$, and let P be a point moving along the side BC. Drop the heights PX, PY from P to the sides AB and AC. Prove that the sum $PX + PY$ remains constant as P moves.



   Solution 1. We can cut triangle ABC into triangles ABP and APC, so

$$\text{Area } ABC = \text{Area } ABP + \text{Area } APC = \frac{AB \cdot PX}{2} + \frac{AC \cdot PY}{2}.$$

   But sides AB and AC have the same length, so this can be rewritten as

$$\frac{AB \cdot PX}{2} + \frac{AB \cdot PY}{2} = \frac{AB}{2}(PX + PY).$$

   Since AB and the area of $\triangle ABC$ are constant, so is $PX + PY$.

   Solution 2. As shown in the right-hand picture above, reflect A and X across BC to form the rhombus $ABA'C$. Then PX' and PY are perpendiculars to parallel sides AC and $A'B$, so they are collinear, and $PX + PY = PX' + PY = X'Y$. This is a constant because parallel lines are always the same distance apart.

3. Two fractions

$$\frac{a}{b} \quad \text{and} \quad \frac{c}{d}$$

   are called *approximately equal* if a, b, c, d are positive integers and

$$\frac{a}{b} - \frac{c}{d} = \frac{1}{bd}.$$

   Prove that given two approximately equal fractions, we can multiply the four numerators and denominators by the same positive integer and then add or subtract 1 from each of them so that the resulting fractions are equal.

   Solution. Given the approximately equal fractions

$$\frac{a}{b} \quad \text{and} \quad \frac{c}{d},$$

   we multiply the four terms by $a + b + c + d$ to get

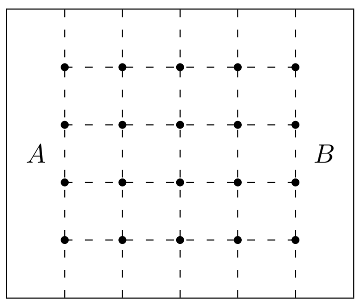
$$\frac{a^2 + ab + ac + ad}{ab + b^2 + bc + bd} \quad \text{and} \quad \frac{ac + bc + c^2 + cd}{ad + bd + cd + d^2}.$$

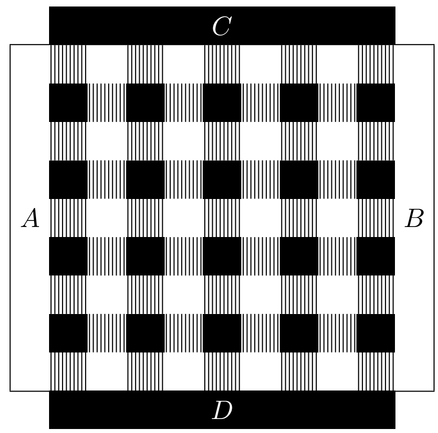
We then adjust each member by 1 by changing the ad terms to bc and vice versa. The resulting fractions can be factorized:

$$\frac{a^2+ab+ac+bc}{ab+b^2+ad+bd} \quad \text{and} \quad \frac{ac+ad+c^2+cd}{bc+bd+cd+d^2}$$
$$\frac{(a+c)(a+b)}{(b+d)(a+b)} \quad \text{and} \quad \frac{(a+c)(c+d)}{(b+d)(c+d)}.$$

The last pair of fractions are visibly equal.

4. In the following maze, each of the dashed segments is randomly colored either black or white. What is the probability that there will exist a path from side A to side B that does not cross any of the black lines?





Solution. In the picture at right, we have distorted the maze without changing its essential structure; the black dots become black squares, and the dashed lines likewise become squares, indicated by stripes, which are randomly black or white. The *nonexistence* of a white path from A to B is equivalent to the existence of an unbroken black wall from C to D; but owing to the symmetry of the maze, these two types of path are equally likely. The desired probability is therefore $1/2$.

5. Prove that there are infinitely many primes p with the following property: there exists a positive integer k such that $2^k - 3$ is divisible by p.

Solution. Suppose that there were only finitely many such primes $p_1, p_2, p_3, \ldots, p_n$. Consider the number

$$N = 2^{(p_1-1)(p_2-1)\cdots(p_n-1)+2} - 3.$$

Clearly, none of the p_i's is 2. Using Fermat's little theorem, we can prove that N is not divisible by any of the p_i:

$$N \equiv 2^{(p_1-1)\cdots(p_i-1)\cdots(p_n-1)} \cdot 4 - 3 \equiv 1 \cdot 4 - 3 \equiv 1 \pmod{p_i}.$$

However, $N > 1$ since the product $(p_1-1)\cdots(p_n-1)$ is at least 1. Therefore N is divisible by a prime other than the p_i, which is a contradiction.

6. Determine whether there exists a polynomial $f(x, y)$ of two variables, with real coefficients, with the following property: A positive integer m is a triangular number if and only if there do *not* exist positive integers x and y such that $f(x, y) = m$.

Remark. A triangular number is one of the numbers $1, 3, 6, 10, \ldots$ of the form $\frac{n^2+n}{2}$, where n is a positive integer.

Solution. The answer is yes.

The difference between the nth and $(n+1)$st triangular numbers is $n + 1$. Therefore, a positive integer m is *not* a triangular number if and only if it has the form

$$\frac{n^2+n}{2} + y$$

where $1 \le y \le n$. Define $x = n - y + 1$; then the conditions $1 \le y$ and $y \le n$ are equivalent to $y \ge 1$ and $x \ge 1$. Conversely, $n = x + y - 1$, so the polynomial

$$f(x,y) = \frac{(x+y-1)^2 + (x+y-1)}{2} + y = \frac{(x+y)(x+y-1)}{2} + y$$

hits exactly the non-triangular numbers as x and y range over positive integers.

7. Let $ABCDE$ be a convex pentagon circumscribed around a circle ω such that $AB \parallel CD$ and $BC \parallel DE$. Locate points X and Y on rays AB and ED, respectively, such that $BX = AB$ and $DY = DE$. Prove that XY is tangent to ω.

Solution. For convenience, let Z be the intersection of sides AB and DE (so we have rhombus $ZBCD$), and let P and Q be the tangency points of ω with AB and DE, respectively. Then it is evident that a necessary condition for XY to touch ω is

$$PX + QY = XY. \tag{1}$$

To see that it is also sufficient, suppose ω and XY are not tangent and consider the concentric circle ω' tangent to XY at T'. Then XP' and YQ' are either both greater or both less than XT' and YT', respectively, and $XT' + YT' = XY$, so (1) is not satisfied.

Let $x = BX$ and $y = DY$; we will think of x and y as varying, keeping XY tangent to ω, while rhombus $ZBCD$ is fixed. Squaring both sides and using the Law of Cosines, we transform (1) to the form

$$(PX + QY)^2 = ZX^2 + ZY^2 - 2ZX \cdot ZY \cdot \cos Z$$
$$(BP + x + DQ + y)^2 = (ZB + x)^2 + (ZD + y)^2 - 2(ZB + x)(ZD + y) \cos Z.$$

It is evident that when this equation is expanded, x^2 and y^2 will cancel, leaving an equation of the form

$$axy + bx + cy + d = 0.$$

We claim that b and c are both zero. If $x \to \infty$ (that is, X runs out along ray ZB), then Y tends toward D, i.e. $y \to 0$. This can only happen if $b = 0$; symmetrically $c = 0$. So we have an equation of the form

$$axy + d = 0.$$

We note that (x, y) satisfies this equation if and only if $(-x, -y)$ does. But $a(-x)(-y) + d = 0$ is exactly the equation obtained by applying the Law of Cosines to the condition that AE is tangent to ω, as the reader can check.