Berkeley Math Circle
Monthly Contest 1
Due October 2, 2012

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 1
by Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Find, with proof, all ways to write 1 as a sum of three fractions, each with numerator 1 and positive integer denominator. (The order of the fractions is irrelevant, so for instance $\frac{1}{2}+\frac{1}{4}+\frac{1}{4}$ is the same as $\frac{1}{4}+\frac{1}{4}+\frac{1}{2}$.)

   Remark. Simply listing all the correct ways to arrange the fractions will only be worth 1 point. To receive full credit, you must explain why the arrangements you have found are the only ones; in other words, why no other selection of three fractions with numerator 1 and positive integer denominator will add to 1.

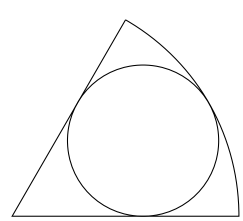
2. Determine whether there exists a number that begins with 2 having the property that, when the 2 is moved to the end, the number is

   (a) doubled;

   (b) tripled.

   Remark. Each part (a and b) requires a separate answer of *yes* or *no*. If you think that the answer is *yes*, you should give an example of such a number. If you think that the answer is *no*, prove rigorously that such a number does not exist.

3. A circle is inscribed in a sector that is one sixth of a circle of radius 6. (That is, the circle is tangent to both segments and the arc forming the sector.) Find, with proof, the radius of the small circle.

   Remark. Be careful in writing your solution. In particular, any statement that you use regarding the symmetry of the diagram must be proved in order to get full credit.

4. Call a positive integer *one-full* if it satisfies the following criteria:

   (a) Every digit is either 0, 1, or 2.

   (b) Out of every two consecutive digits, at least one of them is a 1.

   For $n \geq 2$, calculate the number of n-digit one-full numbers. (Numbers beginning with the digit 0 are not allowed.)

   Remark. Express your answer in terms of n and rigorously prove it.

5. For integers $n \geq 1$, prove that the product

$$3 \cdot 12 \cdot 21 \cdot 30 \cdot \dots \cdot (9n-6)$$

   is divisible by $n!$.

6. Circles j and k, centered at O and P respectively do not intersect. The two tangent rays from O to k meet j at A and B, respectively, and the two tangent rays from P to j meet k at C and D, respectively. Prove that A, B, C, and D are the vertices of a rectangle.

7. In the hold of a pirate ship are ten treasure chests lying on pedestals in a circular arrangement. The captain would like to move each chest clockwise by one pedestal. However, the chests are so heavy that the captain and his assistant can only switch two chests at a time. What is the minimum number of switches needed to accomplish the task?