Random Graphs,
Ramanujan Graphs

Berkeley Math Circle
May 2012

Alon Amit

Ramanujan Graphs

V

Random Graphs

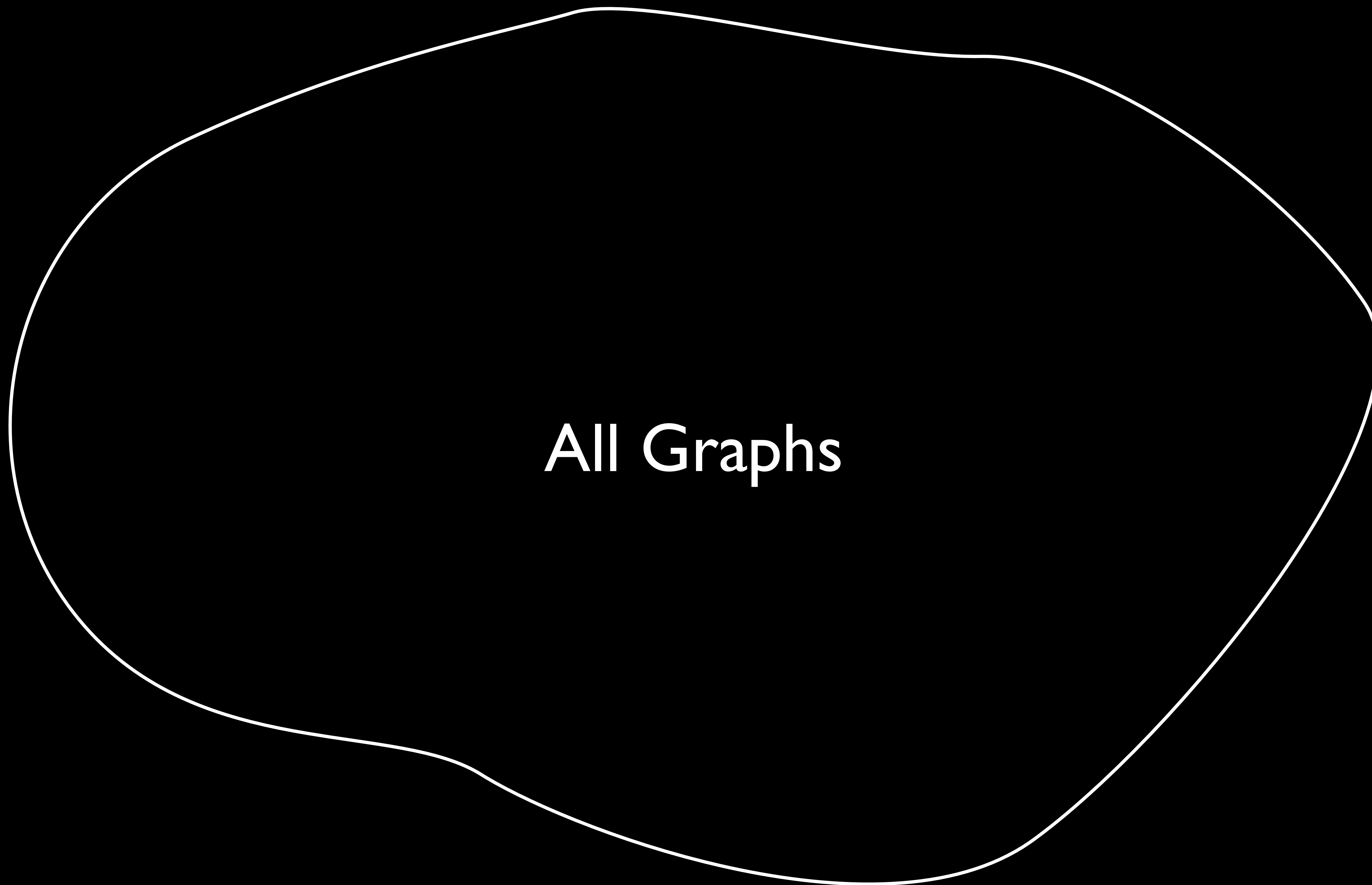
?

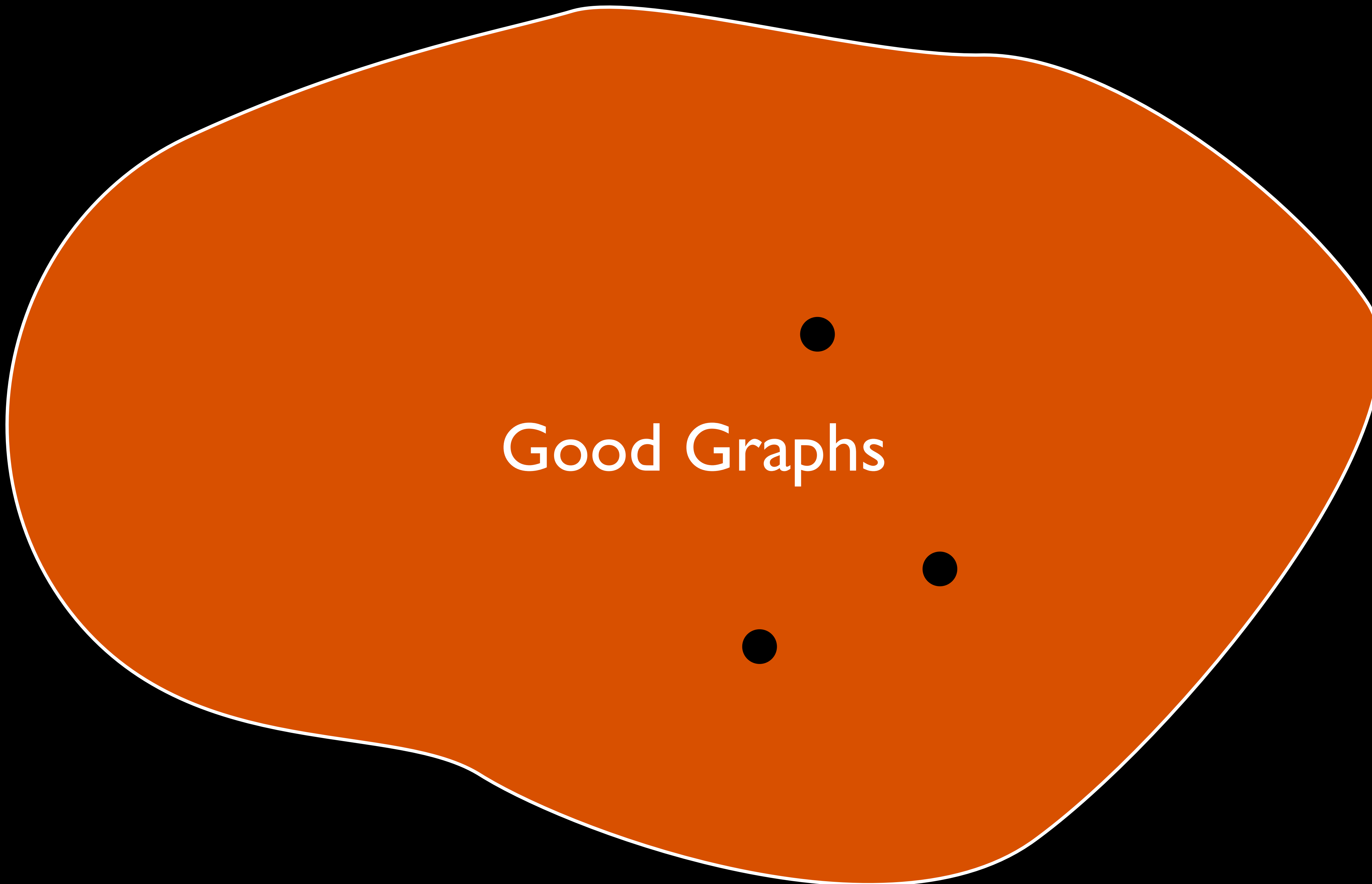
V

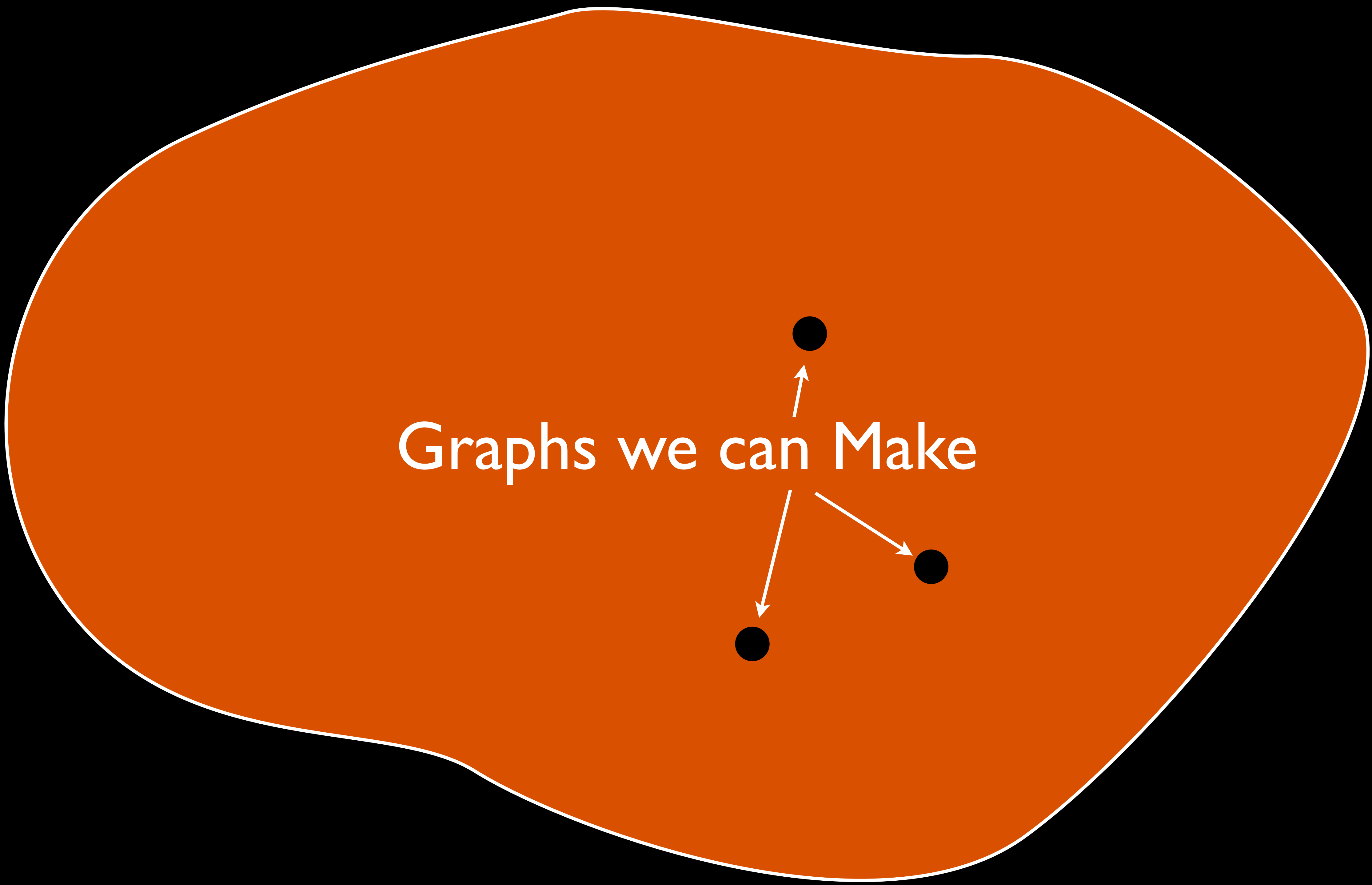
Math isn't Engineering.

Engineering: random stuff is useless.

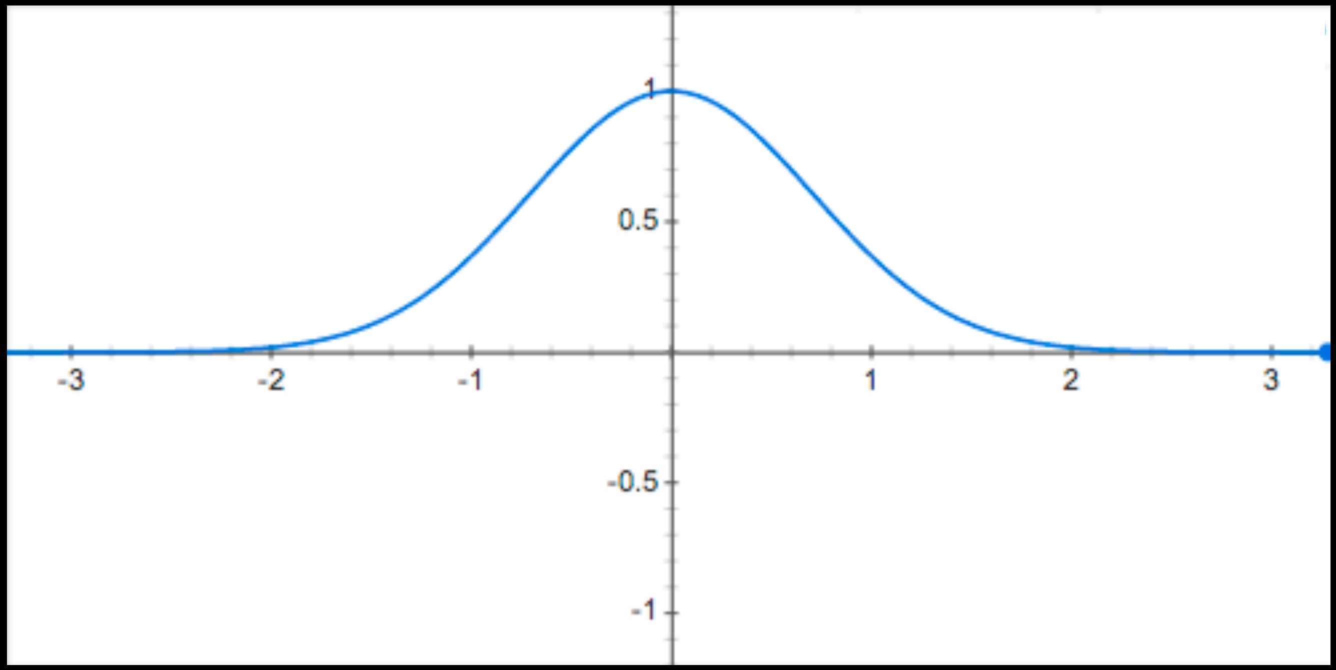
Math: random objects are amazingly useful and incredibly hard to construct explicitly.


All Graphs

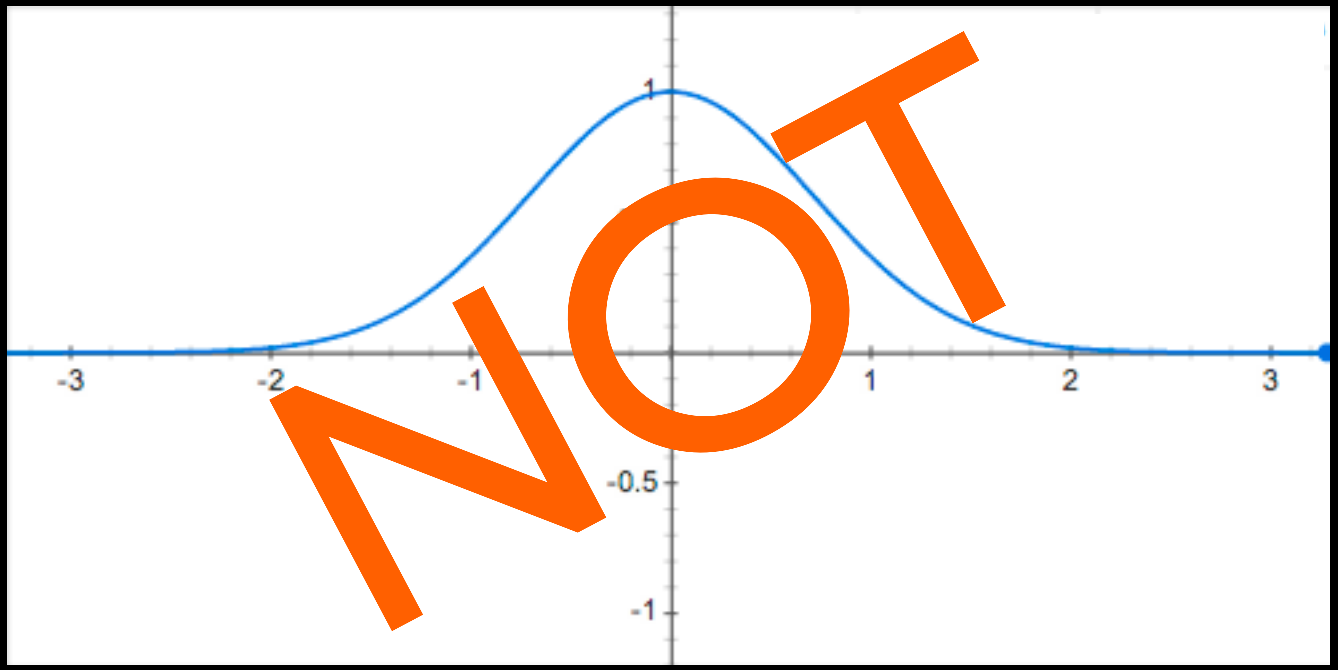

Good Graphs


Graphs we can Make

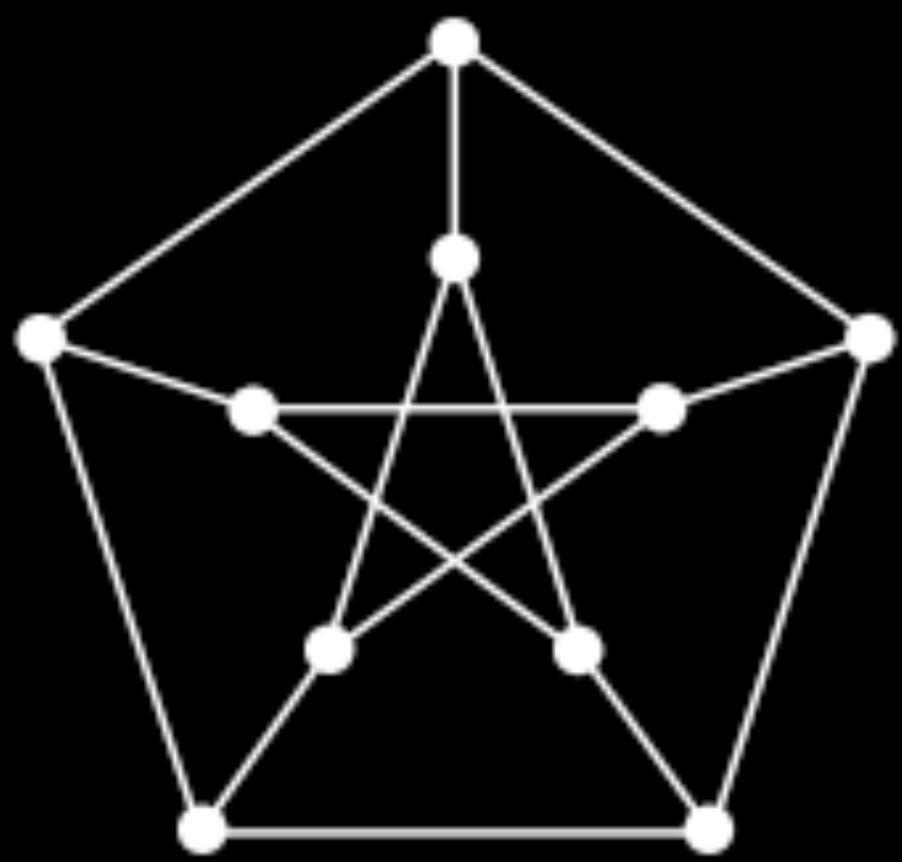
“Graph”



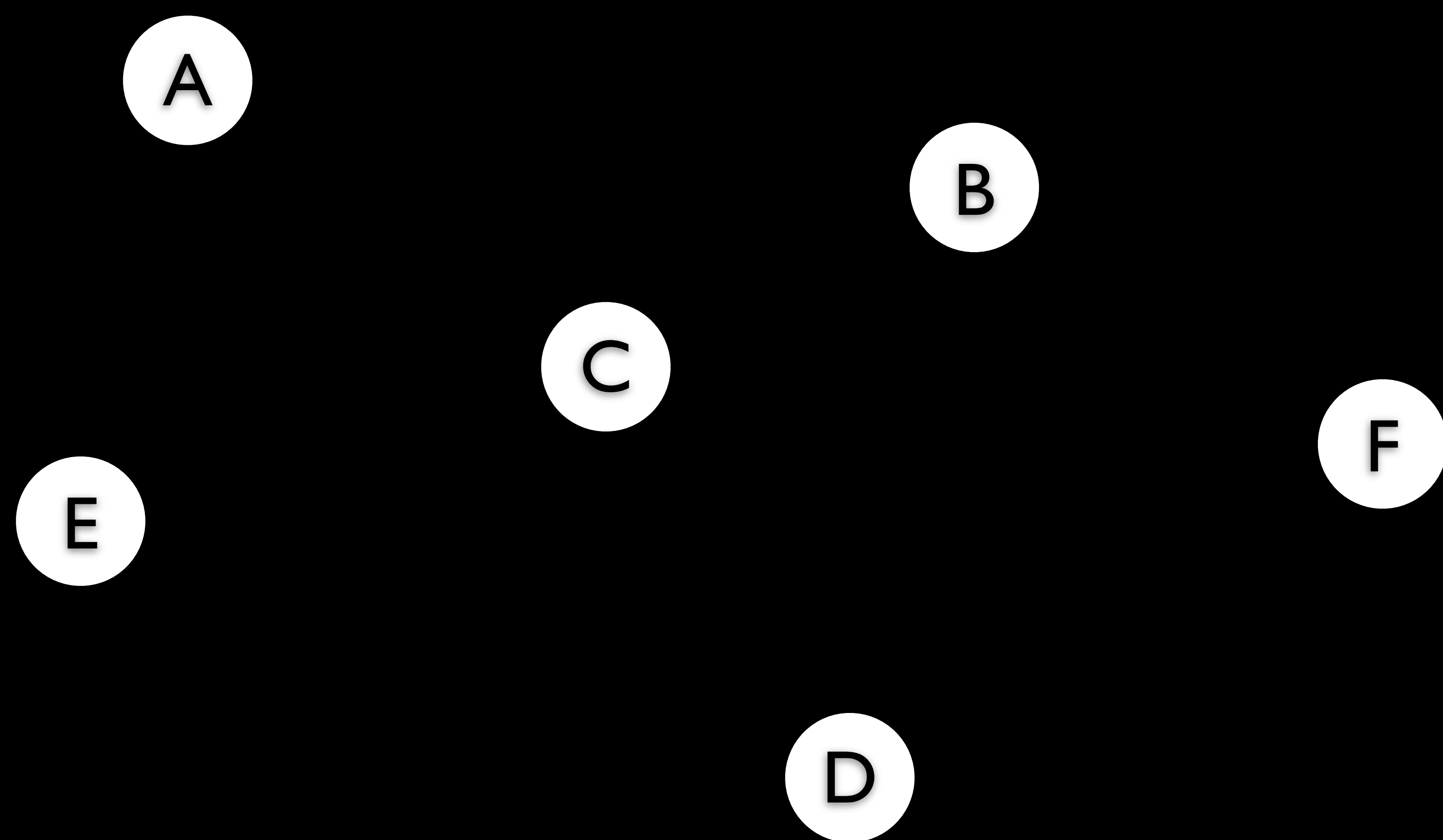
“Graph”



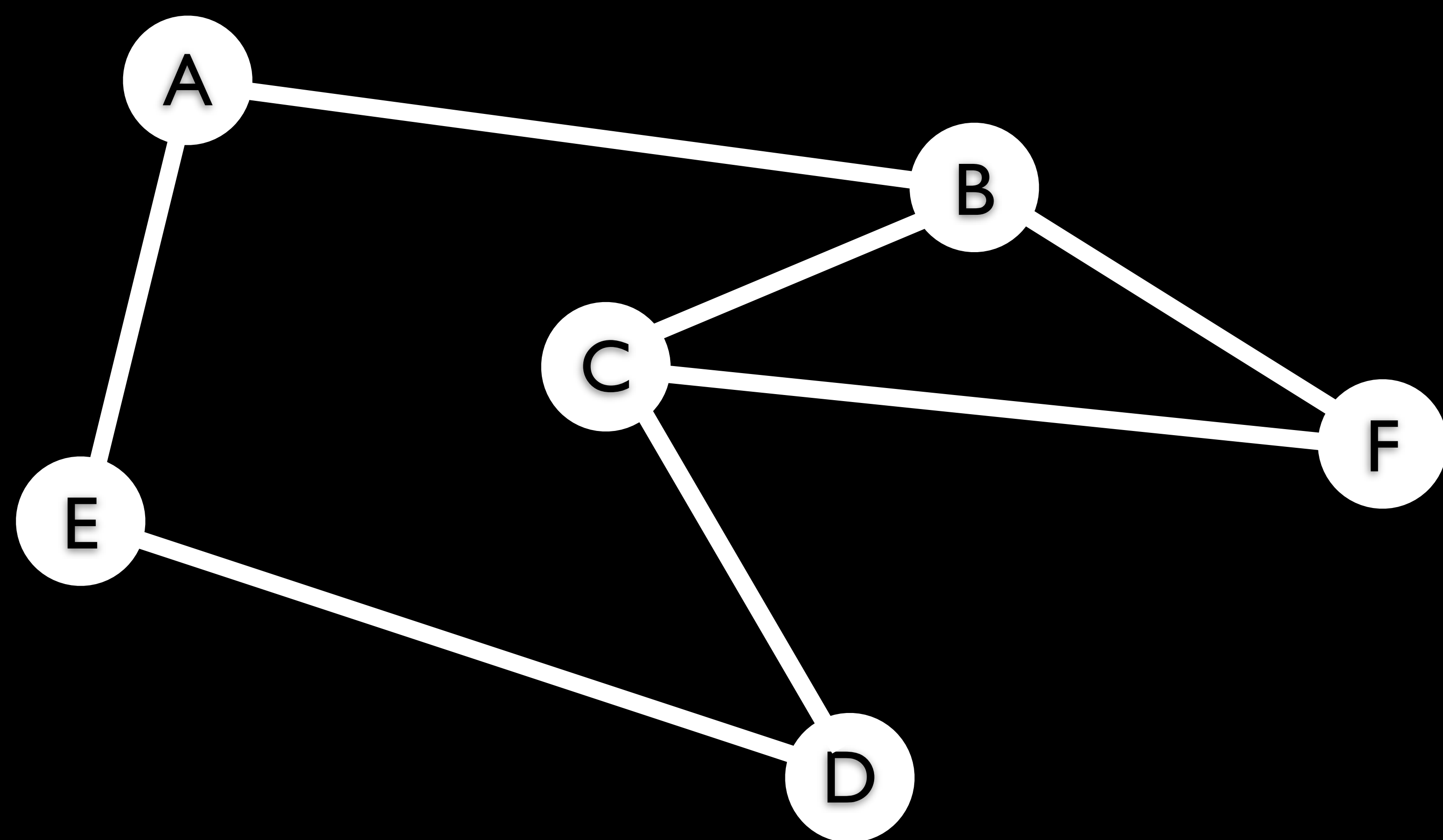
“Graph”



“Graph”



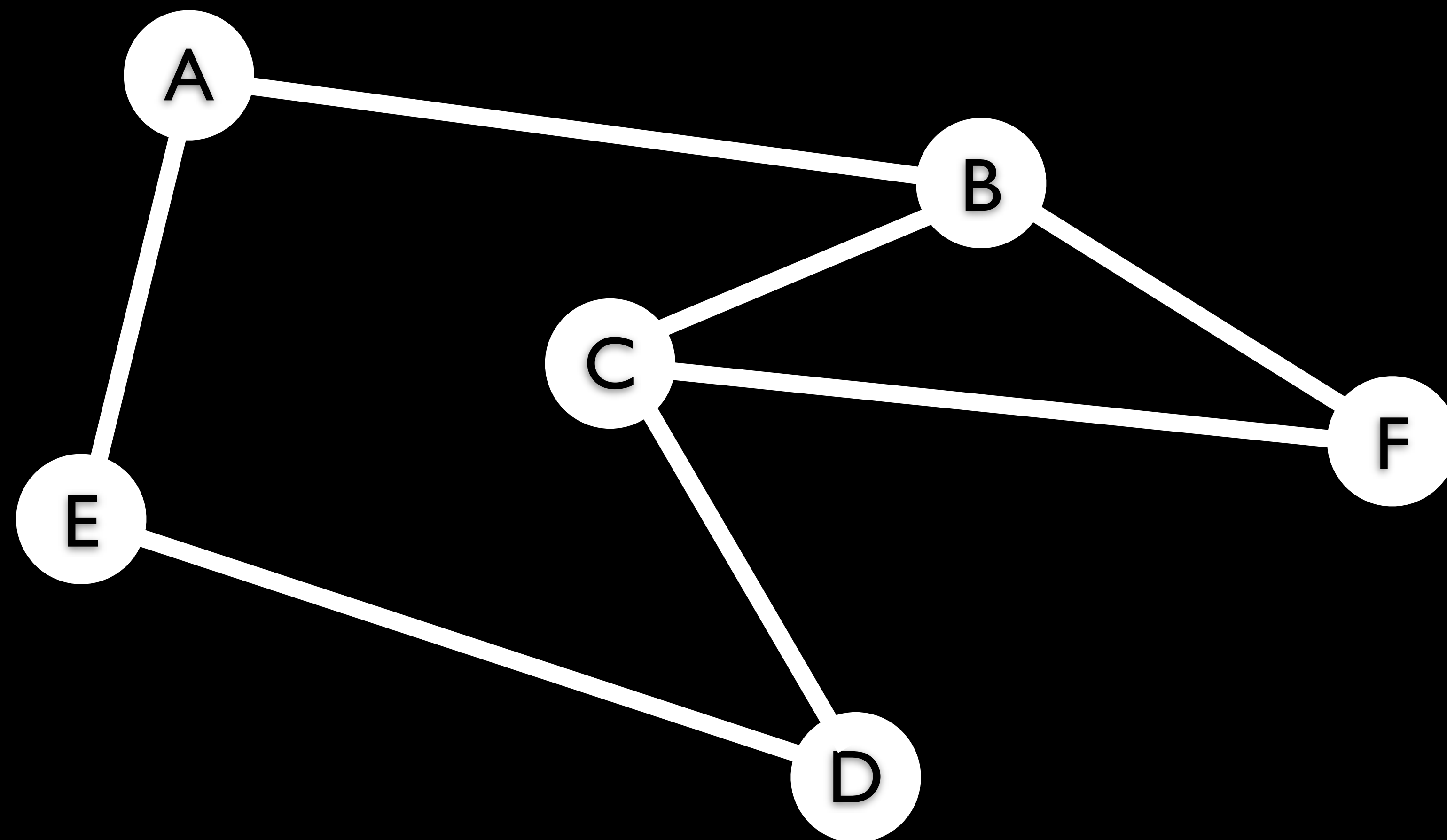
Vertices



Edges

Vertices = $\{A, B, C, D, E, F\}$

Edges = $\{\{A, B\}, \{A, E\}, \{B, C\}, \{C, D\}, \{C, F\}, \{B, F\}\}$



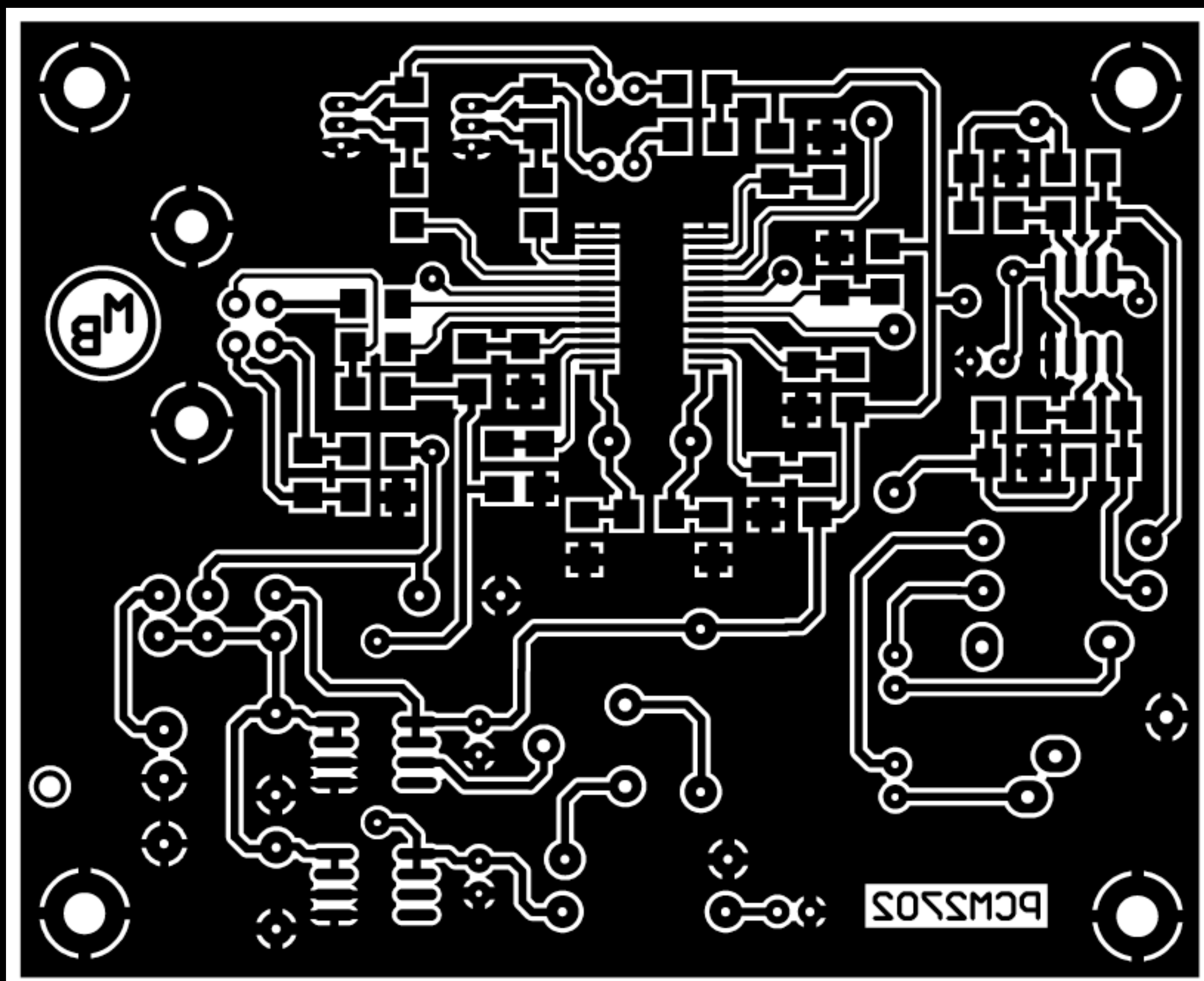
Edges

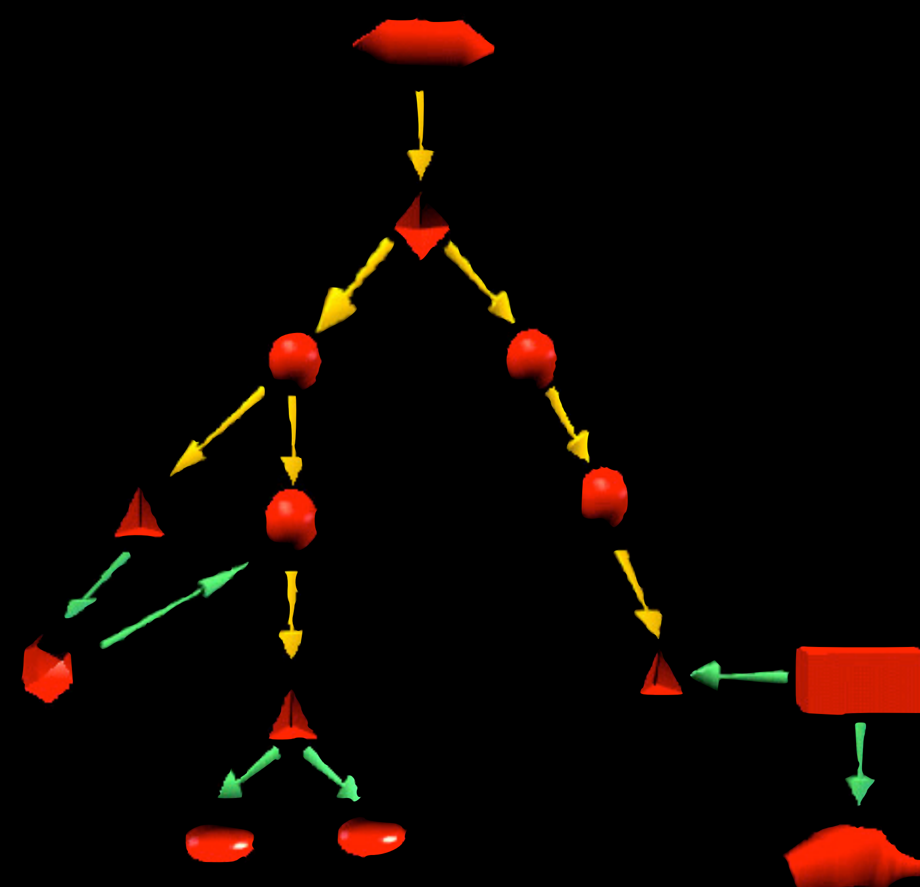
What are Graphs good for?

Lots.

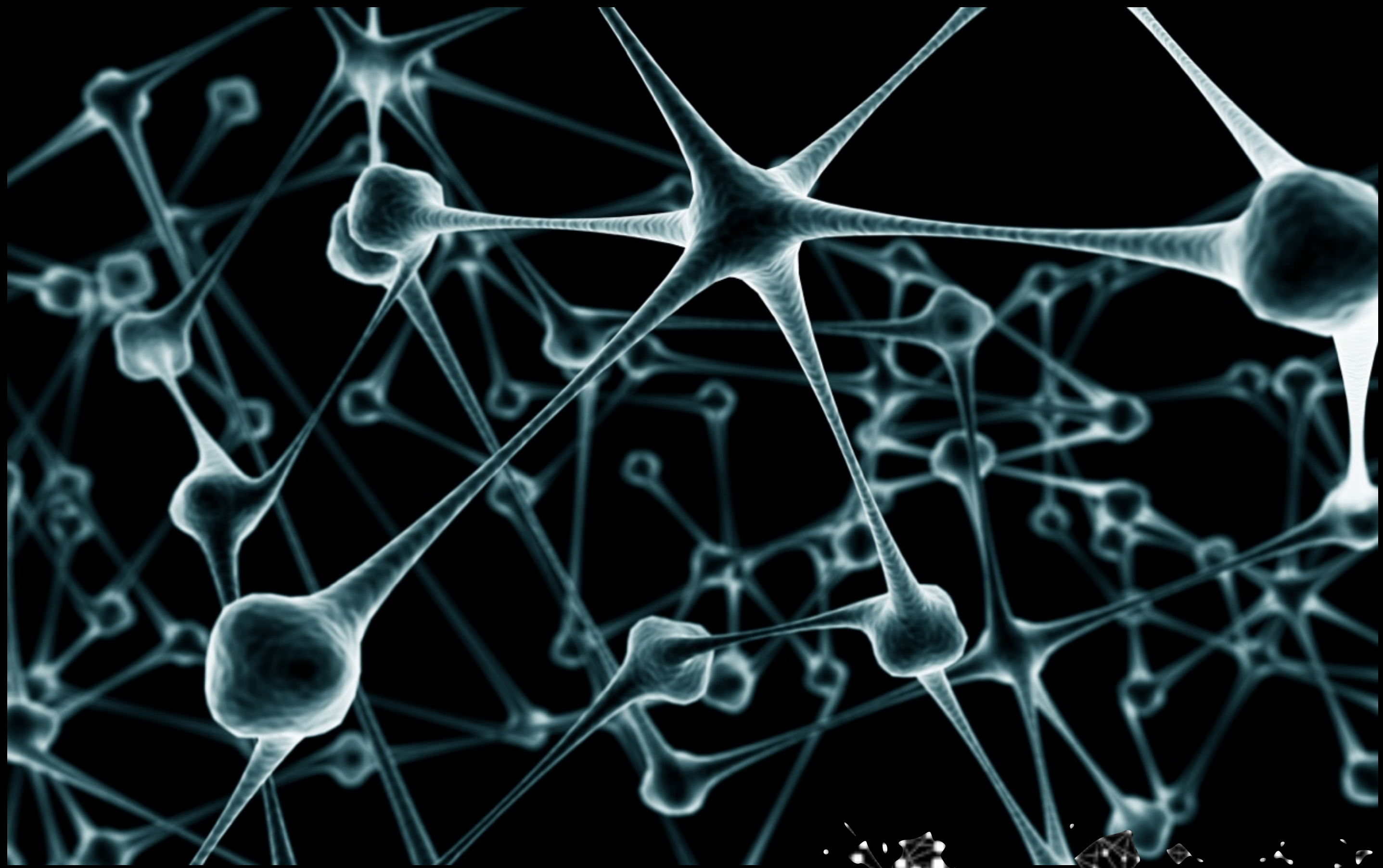
Computer Science

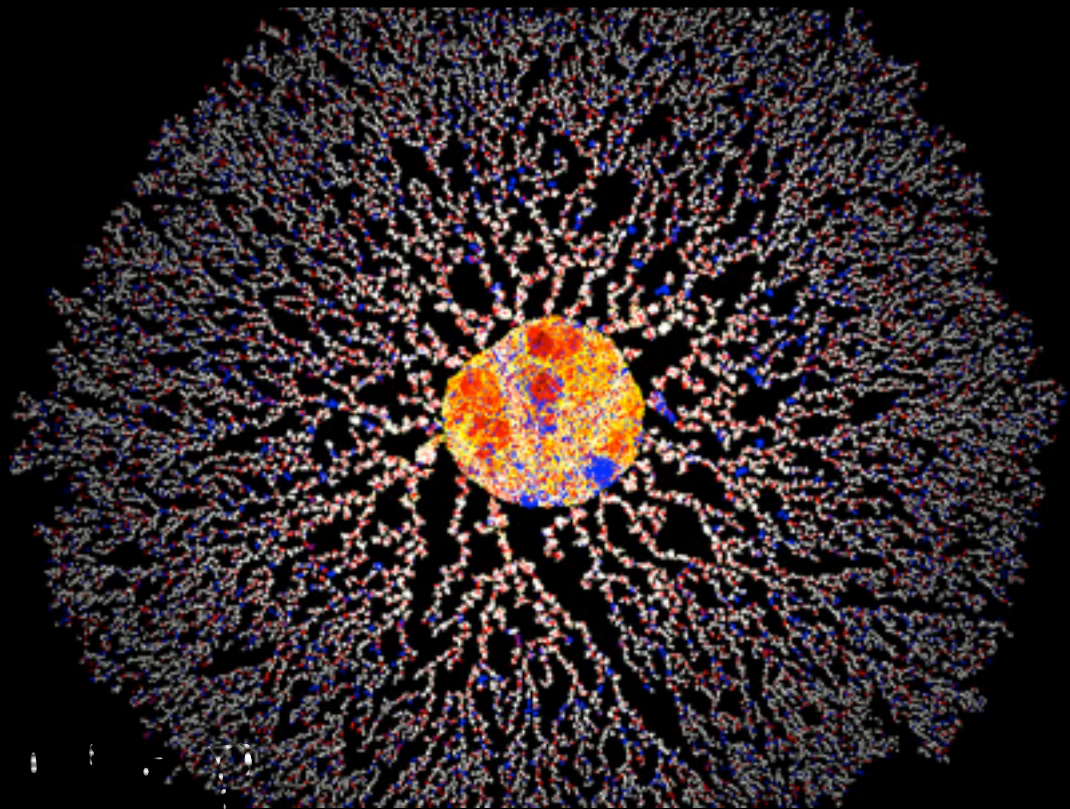
Engineering





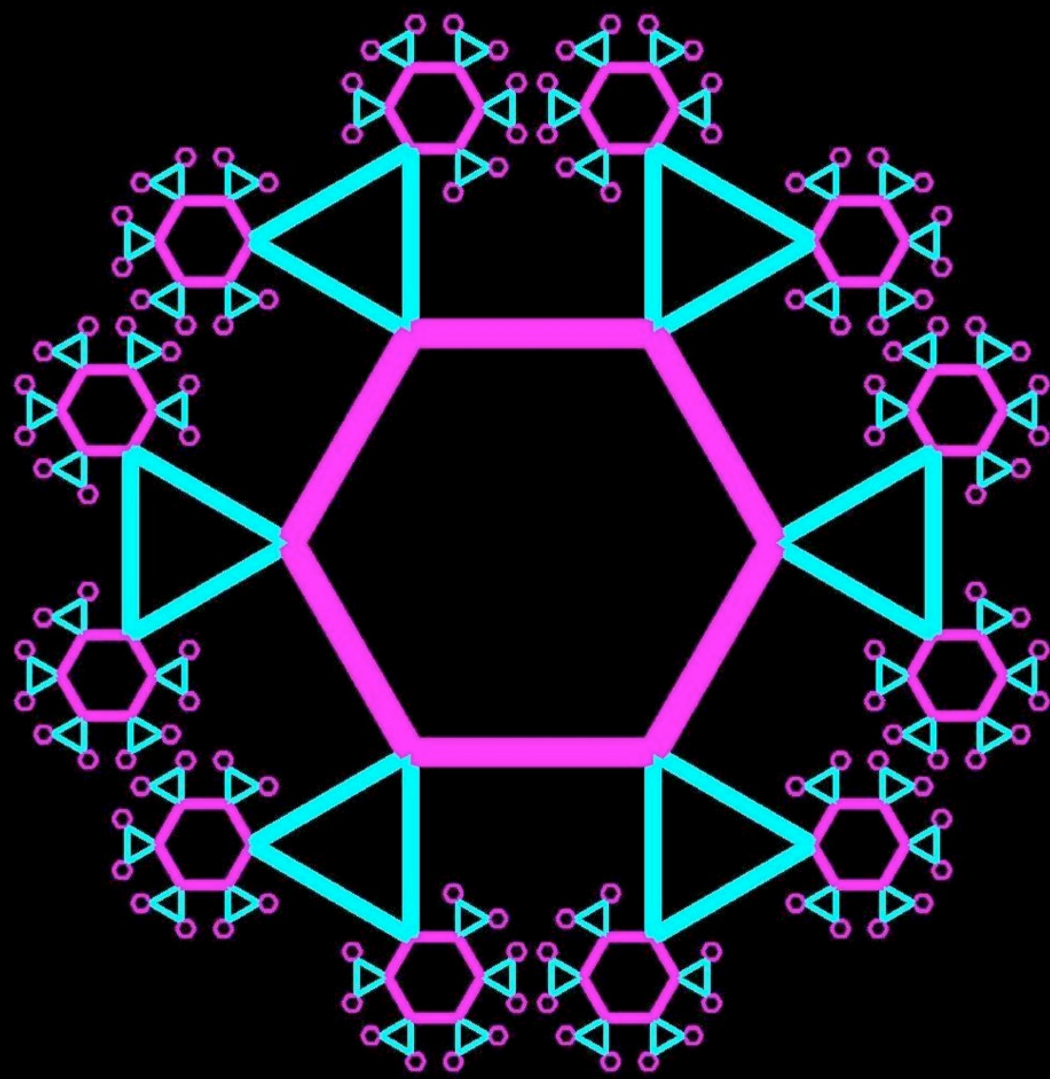
Biology

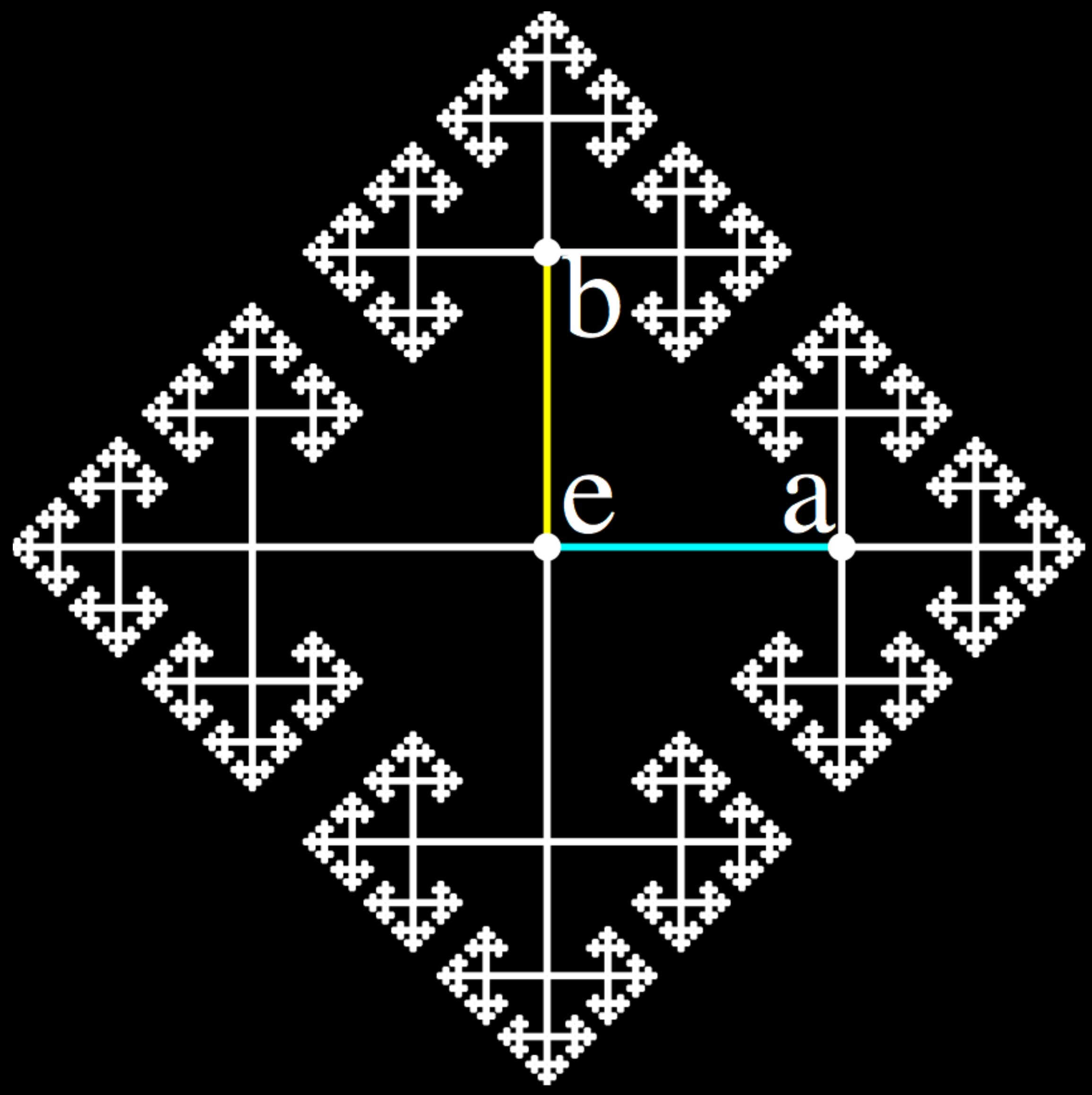




Wednesday, May 9, 12

...and Mathematics.





Random Graphs

Ramsey: Among 6 people there must be 3 friends or 3 strangers.

$$R(3,3) = 6$$

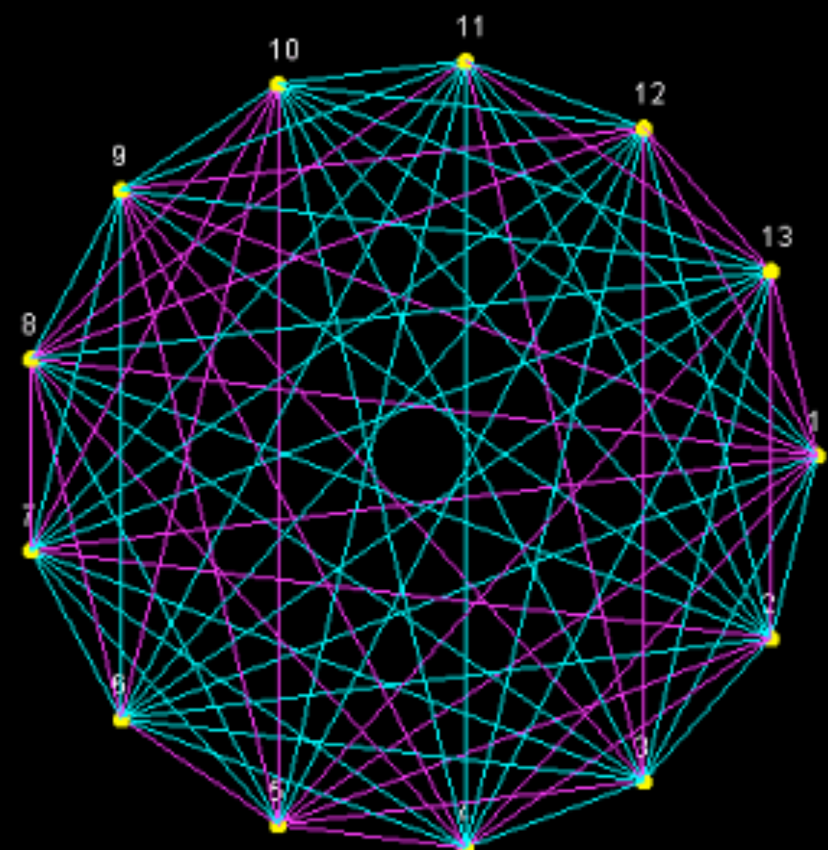
Fact: among 40,000,000,000 people, there must be 20 strangers or 20 mutual friends.

$$R(20, 20) < \binom{38}{19}$$

However: can you arrange for 1,024 people to avoid 20 strangers and 20 mutual friends?

$$R(20, 20) > 1024$$

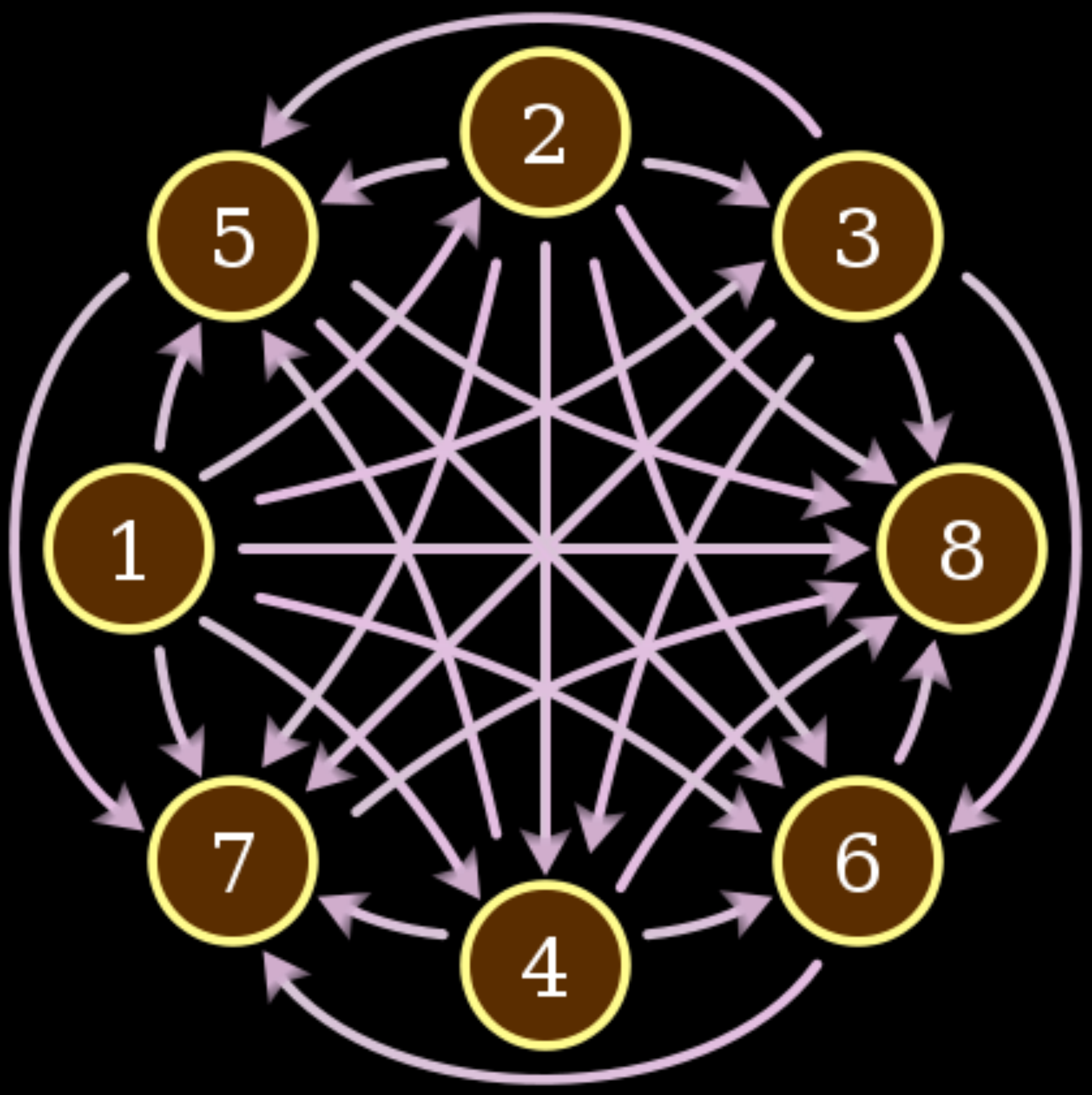
K_n



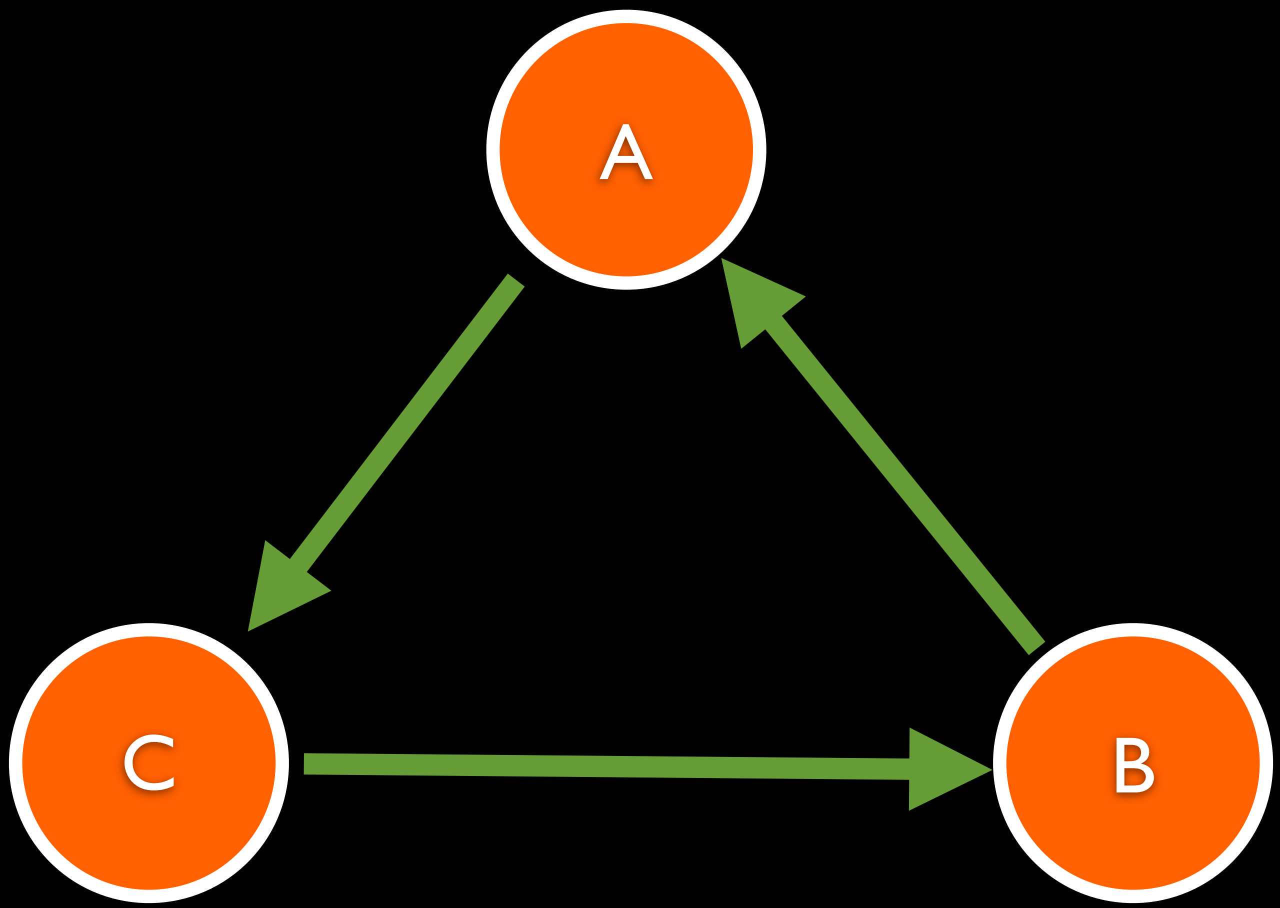
Can we color the edges of K_{1024} in 2 colors so that there's no monochromatic K_{20} ?

Yes. Color at Random.

Tournaments



No Clear Winner



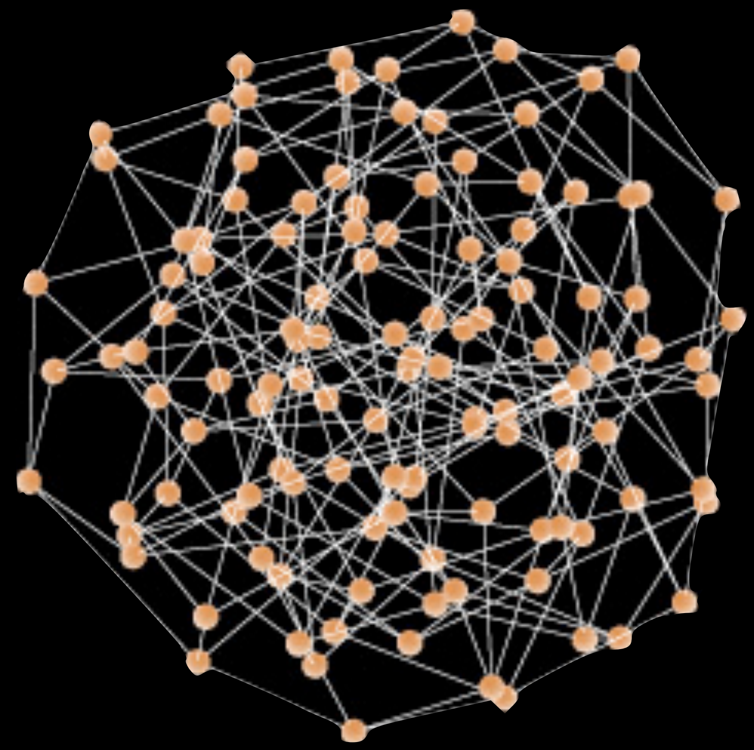
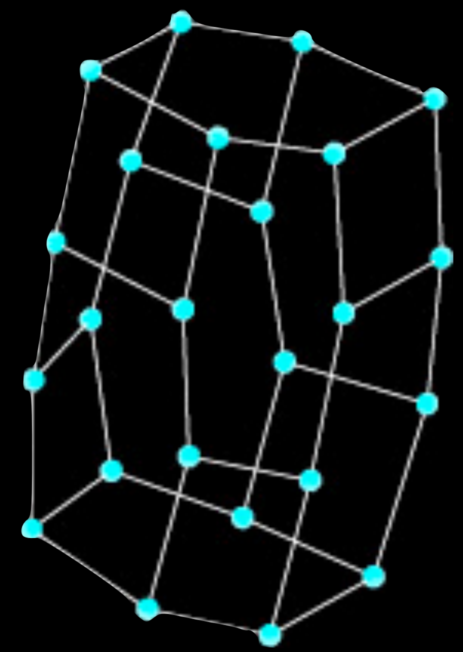
Can you make a tournament in which for every 2 players there's someone who beats them all?

Can we have a tournament in which for every 10 players there's someone who beats them all?

Yes. A Random One.

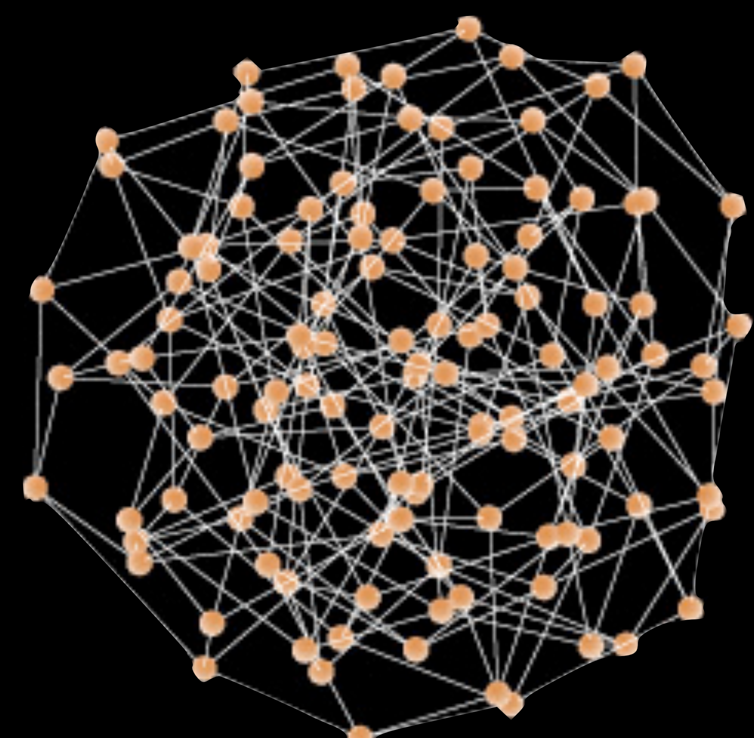
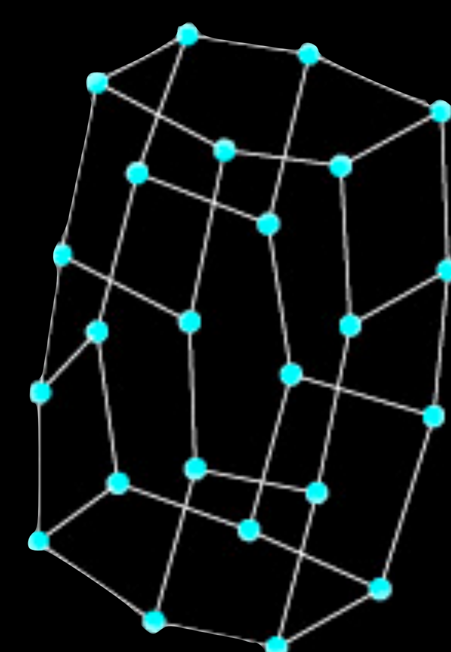
Expander Graphs

- Sparse: few edges.
- Highly connected: Separating more that 20% of the vertices from the rest requires severing many edges.

Expander Graphs

- Communication network design.
- Algorithms: derandomization.

Do Expanders Exist?

Yes. Random graphs are expanders!

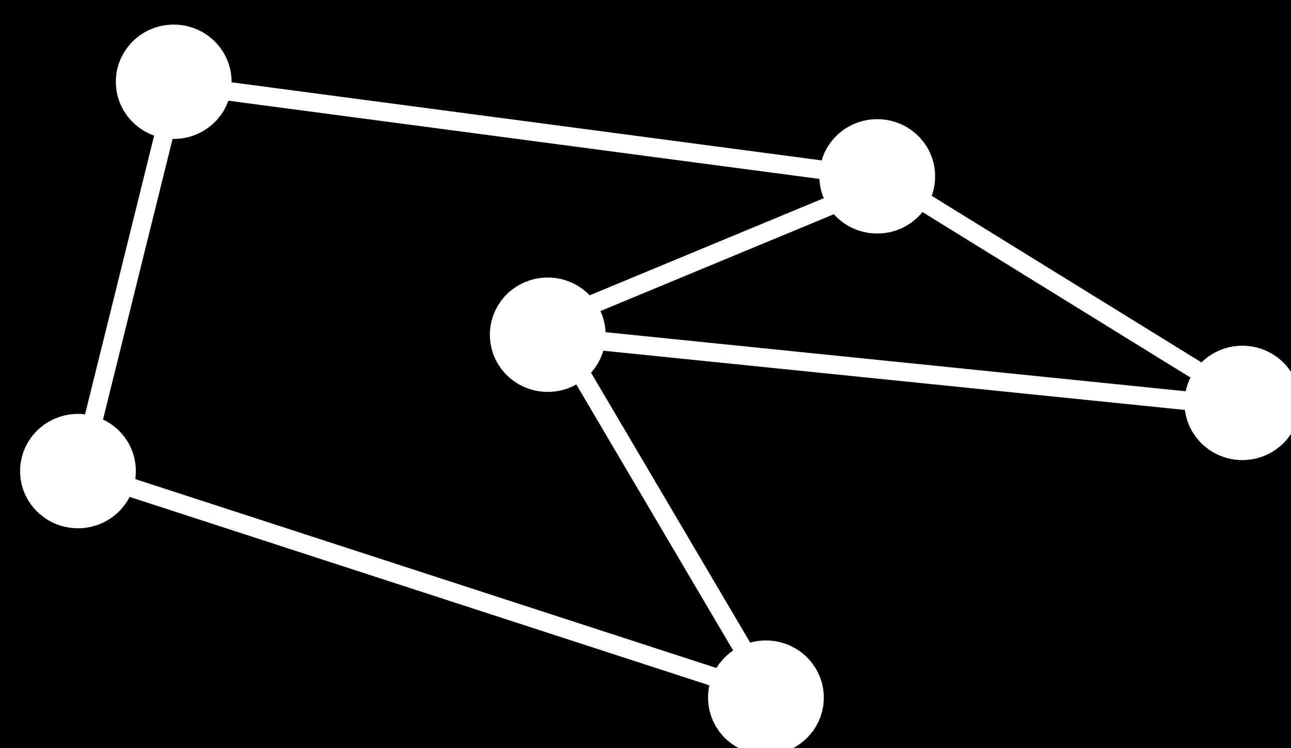
Explicit Construction?



Ramanujan Graphs

Adjacency Matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$



Graph Spectrum

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_1 = 3, \lambda_2 = 2.8, \ldots$$

Graph Spectrum

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_1 = 3, \lambda_2 = 2.8, \ldots$$

Ramanujan Graph

- d-regular: every vertex has d neighbors.
- Connected.
- Spectral gap: $\max_{|\lambda_i|<d} |\lambda_i| \leq 2\sqrt{d-1}$

Nobody knows if random graphs are Ramanujan.

The LPS Construction

- Quotient of the tree $\mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$ by a certain discrete subgroup defined via quaternion algebras.
- Proof of Ramanujan property relies on the proof of the Ramanujan Conjecture for certain cusp forms associated with representations of this group.
- $(p+1)$-regular for prime p.

“Toy” Model

- Define $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$
- Consider the Cayley graph of $\mathrm{SL}_2(q)$ with generators A, B, A^{-1}, B^{-1}
- This graph is 4-regular, connected with $q(q^2-1)$ vertices and “large girth”.

What you Should Remember

- Random graphs have incredible properties that are hard to achieve explicitly.
- Ramanujan Graphs do achieve many such properties.
- Ramanujan Graphs are crazy hard to construct, and the proof that they "work" is even harder.

Acknowledgments

- Prof. Alex Lubotzky
- Prof. Nati Linial