HAVE COLORS – WILL PAINT

Tatiana Shubin tatiana.shubin@sjsu.edu

Imagine that you're completely and utterly free – you've won a long week on a vacation island with no homework, no chores, and no bedtime. You have lots of blank paper, an unlimited supply of different colored paints and you decide to paint whatever wild pattern comes into your mind – stripes, checks, paisley, bird's eye, dog's tooth, herringbone, plaid, polka dots, oh my! Would there be any possible restriction to your exploits? Why should there be any? For example, here's an interesting question:

1. Is it possible to paint every point of a plane with one of three colors so that all three colors are used and every line of the plane consists of points of exactly two colors?

Or how about some others:

2. Is it possible to color each point of a plane with one of two colors in such a way that no two points exactly a unit distance apart are of the same color? What if we use three colors instead? Four colors?

Well, perhaps dealing with the entire plane is too difficult. Let's look at something not as big, say, just one circle.

3. Is it possible to color each point on a circle either red or blue in such a way that no three points of the same color form an isosceles triangle? What if instead of just two colors you can use three different colors? Four colors? 1,000,000 colors?

Some (dis?)similar problems:

A. A.1 Is it possible to split the natural numbers into two sets *A* and *B* such that the sum of two distinct elements of *A* belongs to *B* and vice-versa?

A.2 Suppose that the set of all natural numbers is split into two sets *B* and *R*. We'll call the elements of *B* "blue", and the elements of *R* "red". Must there be integers *x*, *y* such that either all four numbers *x*, *y*, *x* + *y*, and *xy* are red, or all four of them are blue?

A.3 Suppose that natural numbers are partitioned into finitely many pieces:

$$\mathbb{N} = A_1 \cup A_2 \cup \ldots \cup A_n$$

(i.e., every integer is colored by one of n colors). Must there be integers *x*, *y* such that all four numbers *x*, *y*, *x* + *y*, and *xy* are of the same color?[1]

[1] This problem was posed by N. Hidman in 1979. It's still open.

B. B.1 If 5 points lie in a plane so that no 3 points form a straight line, prove that four of the points will always form a convex quadrilateral.

B.2 If 9 points lie in a plane so that no 3 points are collinear, prove that 5 of the points form a convex pentagon.

B.3 If the number of points that lie in the plane is $1+2^{n-2}$ (where $n \geq 3$), and no 3 of them are collinear, ca one always select *n* points so that they form a convex *n*-sided polygon?[2]

All the problems above belong to the part of mathematics called

Ramsey Theory.

Frank Ramsey, an English mathematician, economist and philosopher, proved his famous theorem in 1928. It says that if a number of objects in a set is sufficiently large and each pair of objects has one of a number of relations, then there is always a subset containing a certain number of objects where each pair has the same relation. Ramsey theory is concerned with finding just how large is sufficient.

To be a little more precise, we can look into a problem of finding *Ramsey Numbers.* A slightly different way to state Ramsey's theorem is to say that in any coloring of the edges of a sufficiently large complete graph, one will find monochromatic complete subgraphs. For two colors, Ramsey's theorem states that for any pair of positive integers (r,s), there exists a least positive integer $R(r,s)$ such that for any complete graph with $R(r,s)$ vertices, whose edges are colored *red* or *blue*, there exists either a complete subgraph with r vertices which is entirely blue, or a complete subgraph with s vertices which is entirely red.

Ramsey numbers are very hard to calculate. Only few Ramsey numbers are known so far:

R(3,3) = 6; R(3,4) = 9; R(3,5) = 14; R(3,6) = 18; R(3,7) = 23;
R(3,8)=28; R(3,9) = 36; R(4,4) = 18; R(4,5) = 25.

It is also known that $43 \leq R(5,5) \leq 49$, and $102 \leq R(6,6) \leq 165$, but nobody knows these two numbers exactly. In fact, Erdos used to say that if Aliens invade the Earth and threaten to obliterate it in a year's time unless human beings find R(5,5), w e could possibly avoid the obliteration by putting the world's best minds and fastest computers to the task. But if the aliens demanded that we find R(6,6) within a year, we would have no choice but to launch a pre-emptive attack.

[2] It is known that if there are sufficiently many points than it's possible to find n points forming a convex polygon. It is not known whether or not $1+2^{n-2}$ is a sufficiently large number. This number was conjectured by Erdos in 1934.

Let's go back to problem 3. Let's cut the circle and straighten it up. If three points on the circle formed an isosceles triangle, what would these three points look like on this straight line?

Let's consider the following four statements:

I. If all integers of a number line are colored, each with one of two colors, there must be three *monochromatic* (this means 'of the same color') numbers forming an arithmetic progression.

II. If all *lattice points*[3] of a plane are colored, each with one of two colors, there must be three monochromatic points forming an isosceles right triangle.

III. If all lattice points of a plane are colored, each with one of three colors, there must be three monochromatic points forming an isosceles right triangle.

IV. If all lattice points of a plane are colored, each with one of two colors, there must be four monochromatic points forming a square.

The proofs of these four statements that we've gone through should give you a pretty good idea of how the following celebrated theorem can be proved.

Van der Waerden's Theorem: For any given positive integers r and k, there is some number N such that if the integers $\{1, 2, \ldots, N\}$ are colored, each with one of r different colors, then there are at least k monochromatic integers forming an arithmetic progression.

The least such N is the *Van der Waerden Number* $W(r, k)$. We have just seen that $W(2, 3) \leq 21$. In fact, $W(2, 3) = 9$. It's not too hard to find $W(3, 3) = 27$. The current record for an upper bound belongs to Timothy Gowers (a Fields medallist); he proved that

$$W(r,k) \leq 2^{2^{r^{2^{2^{k+9}}}}}$$

But it is an **open problem** to find the exact values of $W(r, k)$ for most values of r and k, or even to reduce an upper bound (the Gower's bound is way bigger than actual value – check it!).

[3] Points with integer coordinates are called *lattice points*.