Prime Numbers, Factors, and Division Tricks

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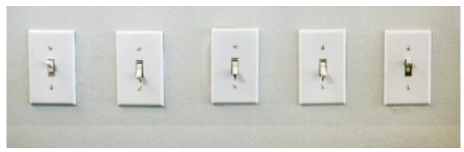
There are 100 light switches on the wall, all turned off. A hundred toddlers come by. The first toddler flips every switch. Then the second toddler flips just switches 2, 4, 6, 8, ... etc. Then the third toddler flips switches 3, 6, 9, 12, ... etc. This pattern continues until finally the 100th toddler flips just switch number 100. How may lights are turned on at the end?



1 Divisibility Rules

1. In base 10, you can tell if a number is even based on whether or not its last digit is even. State and prove a condition (involving the representation of a number) that allows you to determine whether a number is odd or even
   (a) in the base 3 number system
   (b) in the base n number system
2. Find and prove a divisibility rule in base 7 arithmetic that is analogous to the rule (in ordinary base 10 arithmetic) for divisibility by 9. See if you can find other divisibility rules in base 7 arithmetic that are similar to rules for base 10.

2 Factors

> **Fundamental Theorem of Arithmetic:**
> Any natural number other than 1 can be written as a product of prime factors. Except for the order of the factors, this can only be done in one way.

3. Is it true that if a number is divisible by 4 and by 3, then it must be divisible by $4 \cdot 3 = 12$?
4. Is it true that if a number is divisible by 4 and by 6, then it must be divisible by $4 \cdot 6 = 24$?
5. Prove that the product of any three consecutive natural numbers is divisible by 6.
6. How many consecutive numbers do you need to guarantee that their product is divisible by 30? by 120?
7. How many digits are in the product $4^5 \cdot 5^{10}$? (AMC-8 2011)
8. How many zeros are at the end of 100! ? (100! means $100 \cdot 99 \cdot 98 \cdot 97 \cdots 3 \cdot 2 \cdot 1$)
9. Find the smallest natural number n such that $n!$ is divisible by 990? What about 560?
10. For some number n, can the number $n!$ have exactly 5 zeros at the end?

11. Sabrina multiplied two 2-digit numbers together on the blackboard. Then she changed all numbers to letters (different digits changed to diffferent letters, equal digits to equal letters. She got : $AB \cdot CD = EEFF$. Prove that Sabrina made a mistake somewhere.

12. The numbers a and b satisfy $56a = 65b$. Prove that $a + b$ is composite (not prime).

3 Classic Proofs

13. How many prime numbers are there? Prove it!

14. Prove that $\sqrt{3}$ cannot be written as a fraction of natural numbers: $\dfrac{a}{b}$.

4 Just for Fun

15. Find the last digit of the number 2012^{2012}

16. Find the remainder when 3^{2012} is divided by 7.

17. Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

18. Prove that the number $n^3 + 3n$ is divisible by 3 for any natural number n.

19. Suppose that a, b, and c are integers such that $a^2 + b^2 = c^2$. Prove that at least one of a, b, and c is divisible by 3.

Many of these problems are from *Mathematics Circles: the Russian Experience* by Fromkin, Genkin, and Itenberg