Fancier Counting I: Compositions and Partitions

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The idea here is to count the number of ways of splitting up a number into parts. For now we'll use only positive integers.

There are two basic kinds: one where order matters, called a "composition", so for instance the compositions of 3 are 3, 2+1, 1+2, and 1+1+1. They've been sorted here to make it easier to check that you haven't left one out. We see there are four compositions of 3. You might want to think about other ways of organizing this collection.

On the other hand, "partitions" are where order doesn't matter. So 2+1 and 1+2 are the same partition of 3. Thus, 3 has 3 partitions: 3, 2+1, and 1+1+1.

Problem 1. How many compositions does 4 have? Can we organize a list of how many compositions each number has, say from 1 through 10?

Problem 2. What if we then organize them by how many parts they have? Can you predict how many compositions of 10 will have 1 part? 9 parts? 4 parts?

Problem 3. What if we organize them by how big the largest part is? How many compositions of 10 have largest part 1? Largest part 2? Largest part 4?

Problem 4. A *partition* of a number is like a composition, but order doesn't matter, so 1+1+2 and 1+2+1 are two different ways to write the same partition of 4. How many partitions does 4 have?

Problem 5. Organize a table of partitions, at least including all the partitions of 1, 2, 3, 4, and 5. How can you organize your table?

Problem 6. One way to organize your table is based on the smallest partition. How many partitions of 5 have a smallest part of 2? Or, maybe an easier way to uncover a pattern, how many partitions of 5 have no parts smaller than 2?

Problem 7. How many partitions of 5 into 2 parts are there? Can you uncover a pattern for this, so that you could predict how many partitions of 10 into 4 parts there are?

Problem 8. How many partitions of a number are there if all the parts must be even?

Problem 9. How many partitions are there when each part has two copies? Is it easier to count the situations with at least two copies of each part, or exactly two?

Sometimes it's very helpful to be able to make a picture of whatever problem you're working on. For partitions, one useful picture is the “Ferrers diagram” where you list the parts in decreasing order and use dots to represent the numbers. For example, all the partitions of 4 will look like this:

4	3+1	2+2	2+1+1	1+1+1+1
****	***	**	**	*
	*	**	*	*
			*	*
				*

Problem 10. There's a relationship here between 4 and 1+1+1+1: you switch the rows and columns with each other. This operation is called a “transposition” and partitions that are transposes of each other are sometimes called “conjugates”. What is the conjugate of the partition 4 + 2 + 1?

Problem 11. How many partitions of each number are into distinct odd parts? For instance, for 4 we see that 3+1 is the only such partition.

Problem 12. How many partitions of each number are self-conjugate? For instance, for 4 we see that 2+2 is the only example.

Problem 13. Count how many 1s appear in all the partitions of each number. Compare with the total number of different parts that appear in all the partitions (so for instance, for 4 you'd count 2+1+1 as having 2 different parts and 1+1+1+1 as having 1).

Problem 14. Add up the largest part in all the partitions of each number.

Problem 15. When is the number of partitions into an odd number of different parts the same as the number of partitions into an even number of different parts?

Problem 16. And, one final bonus problem: How many ways are there to make change for a dollar?