Fancier Counting III:
BAMO Practice

Berkeley Math Circle, February 21, 2011

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Problem 1. (BAMO 2010) We write $\{a, b, c\}$ for the set of three different positive integers a, b, and c. By choosing some or all of the numbers a, b, and c, we can form ????? nonempty subsets of $\{a, b, c\}$. We can then calculate the sum of the elements of each subset. For example, for the set {4, 7, 42} we will find sums of ???? for its subsets. Since 7, 11, and 53 are prime, the set {4, 7, 42} has exactly three subsets whose sums are prime. (Recall that prime numbers are numbers with exactly two different factors, 1 and themselves. In particular, the number 1 is *not* prime.)

What is the largest possible number of subsets with prime sums that a set of three different positive integers can have? Give an example of a set $\{a, b, c\}$ that has that number of subsets with prime sums, and explain why no other three-element subset could have more.

How can we generalize or extend this problem?

Problem 2. (BAMO 2010) A *clue* "k digits, sum is n" gives a number k and the sum of k distinct, nonzero digits. An *answer* for that clue consists of k digits with sum n. For example, the clue "Three digits, sum is 23" has only one answer: 6, 8, 9. The clue "Three digits, sum is 8" has two answers: 1, 3, 4 and 1, 2, 5.

If the clue "Four digits, sum is n" has the largest number of answers for any four-digit clue, then what is the value of n? How many answers does this clue have? Explain why no other four-digit clue can have more answers.

Problem 3. (BAMO 2009) A square grid of 16 dots contains the corners of ??? 1 by 1 squares, ??? 2 by 2 squares, and one 3 by 3 square, for a total of ??? squares whose sides are parallel to the sides of the grid. What is the smallest number of dots you can remove so that, after removing those dots, each of the ??? squares is missing at least one corner?

Justify your answer by showing both that the number of dots you claim is sufficient and by explaining why no smaller number of dots will work.

How can we generalize or extend this problem?

Problem 4. (BAMO 2008) What is the largest number of S-tetrominoes that can fit in a 9 by 9 grid?

Problem 5. (BAMO 2007) A 15-inch-long stick has four marks on it, dividing it into five segments of length 1, 2, 3, 4, and 5 inches (although not necessarily in that order) to make a "ruler". For example, you might put the segments in the order 2, 3, 5, 1, 4. Using this ruler you could measure 8 inches (3+5) and 11 inches (2+3+5+1), but there's no way you could measure 12 inches. Prove that it is impossible to make such a ruler that would be able to measure every length from 1 through 15 inches.