Is Plane Geometry Plain?
Are There "Magical" Rulers and Protractors?

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Berkeley Math Circle, April 3, 10, 17, May 1 2012

1. Reconstructing a Parallelogram

1.1. **Initial food for thought.** Try the following problem from the Bay Area Math Olympiad 2012:

> **Problem 1. (BAMO 8, 2012)** Laura won the local math olympiad and was awarded a **"magical" ruler**. With it, she can draw (as usual) lines in the plane, and she can also measure segments and replicate them anywhere in the plane; but she can also divide a segment into as many equal parts as she wishes; for instance, she can divide any segment into 17 equal parts. Laura drew a parallelogram $ABCD$ and decided to try out her magical ruler; with it, she found the midpoint M of side CD, and she extended side CB beyond B to point N so that segments CB and BN were equal in length. Unfortunately, her mischievous little brother came along and erased everything on Laura's picture except for points A, M and N. Using Laura's magical ruler, help her reconstruct the original parallelogram $ABCD$: write down the steps that she needs to follow and prove why this will lead to reconstructing the original parallelogram $ABCD$.

Along the way, you will probably need to use similar triangles, regardless of what solution you come up with. Hence, you will need to know how to establish that two triangles are similar, and then how to use this similarity. There are several criteria for establishing similarity between triangles. Below, we list the three most popular such criteria.

Theorem 1. (Criteria for similarity of $\triangle$'s.) Two triangles are similar if

(a) **(AA)** two of their angles are correspondingly equal;
(b) **(SSS)** their sides are correspondingly proportionate;
(c) **(SAS)** two of their sides are correspondingly proportionate, and the corresponding angles between those sides are equal.

1.2. **The skeleton of construction problems.** When solving any construction problem, mathematicians follow the steps:

(1) **Discuss.** Start with a (final, complete) picture of what you are trying to construct. Discover and discuss aspects of the drawing that are relevant to the given problem.

(2) **Construct.** Write an algorithm that constructs, step-by-step, the required object(s).

(3) **Prove.** Prove that your algorithm works: explain why it gives what the problem requires.

(4) **Analyze.** Research how many answers are possible given different initial configurations.

Did you follow the above steps in your solution to Problem 1? If not, go back and try to redo your solution in order to make it complete.

1.3. **Classic geometric tools.** However, if you had asked a mathematician from the past to solve this problem for you, he/she would NOT have had a magical ruler! The only tools available to ancient mathematicians would be:

- **a straightedge** (with no markings on it!) and **compass**.

You may be unaware of how powerful these two tools are and what a variety of constructions they can create! In fact, the problem which our ancient mathematician that have tried to solve instead can been stated simply like this:

Problem 1′. (Parallelogram)[1] M is the midpoint of side CD in the parallelogram $ABCD$, and point N is symmetric to C with respect to point B. Reconstruct the parallelogram $ABCD$ if you know only the position of points A, M, and N.

Can we solve this problem without a magical ruler, using only a straightedge and compass?

2. Conquering the "Magical" Ruler

2.1. **Basic geometric constructions with segments.** Using only a straightedge and compass, show that it is possible to construct the objects below. Give an algorithm for each construction and prove that it does what it is supposed to do.

Exercise 1. Given a segment AB, find its *midpoint* M.

Exercise 2. Given a point A on a line l, *erect a perpendicular* AB to l.

Exercise 3. Given line l and point A not on l, *drop a perpendicular* from A to l.

Exercise 4. Given line l and point A not on l, draw a line n through A *parallel* to l.

Problem 2. Given a segment AB, find two points M and N on it that trisect AB into 3 equal parts: $AM = MN = NB$. More generally, prove that you can divide any segment AB into as many equal parts as you wish.

Conclude that whatever you can do with a magical ruler, you can also do with just a straightedge and compass. Thus, we do NOT need a "magical" ruler! How about the converse: is there something you can do with a straightedge and compass that you cannot do with the magical ruler? Which is stronger: the "magical" ruler or the straightedge and compass?

2.2. **Re-constructing the parallelogram again!** Now that you know how to go by **without** a "magical" ruler, you can go back and finish Problem 1′.

Exercise 5. Reconstruct the parallelogram in Problem 1′ via a straightedge and compass and

(a) by using the midpoint L of MN.
(b) by using the intersections point P, Q, and O of the diagonal BD with AM, MN, and AC.
(c) discovering your own useful objects different from parts (a) and (b) above.

Exercise 6. Analyze the number of possible solutions in Problem 1′. In particular, show that there is a unique parallelogram whenever the given points form $\triangle AMN$, and that there is no solution if the given points A, M, and N lie on a line.

[1] This problem, as well as the other construction problems, in this handout are from "Mathematical Olympiads," part II, by Budurov and Serafimov, Sofia, Bulgaria, 1985.

3. Reconstructing a Square

Let's try the following problem using only the two classic tools: a straightedge and compass.

Problem 3. (Square) Reconstruct a square if given the sum of the lengths of a side and a diagonal (i.e., given $s = a + d$).

Along the way, you will discover that you need to draw certain special angles. So, we go back and continue with our basic geometric constructions.

Exercise 7. Given angle $\angle ABC$, draw its *angle bisector* BL, i.e., a ray $BL^{\rightarrow}$ s.t. $\angle ABL = \angle CBL$.

Exercise 8. Draw a right angle, a $45°$-angle, and a $22.5°$-angle.

In addition, you will probably need to use the following basic geometric facts:

Theorem 2. (Angles all over)

(a) **($180°$ Sum in $\triangle$)** The angles in a triangle add up to $180°$.
(b) **(Exterior Angle Theorem)** An exterior angle to a triangle equals the sum of the two (remote) interior angles.
(c) **(Parallel Lines & Equal Angles.)** Two parallel lines and a transversal form eight angles, which can be grouped into two groups of four each: the four angles in each group are equal.

Exercise 9. Analyze the number of possible solutions in Problem 2.

4. More Advanced Geometric Constructions as a Long-Term Project

In the following problems, make sure that you go over the four steps in construction problems: discussion, construction, proof, and analysis of possible number of solutions.

Problem 4. (Quadrilateral) Reconstruct a quadrilateral if given

(a) the midpoints of three of its sides and the intersection point of the diagonals and if it is known that the quadrilateral is convex.
(b) the lengths of its four sides and if one of the diagonals is the angle bisector of one of the quadrilateral's angles.

Problem 5. (Rhombus) Reconstruct a rhombus if given

(a) the lengths of a diagonal and an altitude h.
(b) the difference between its diagonals, i.e., $d_1 - d_2 = m$, and one of its angles α.
(c) the sum of its diagonals, i.e., $d_1 + d_2 = m$, and one of its acute angles α.

Problem 6. (Trapezoid) Reconstruct a trapezoid $ABCD$ if given

(a) the base $AB = a$, the altitude $DE = h$ (E lies on AB), and the diagonals $AC = d_1$ and $BD = d_2$.
(b) the sides $AD = b$ and $BC = c$, the altitude $DE = h$ (E lies on AB), and the diagonal $DB = d$.
(c) the sum of the bases $AB + CD = s$, $\angle BAD = \alpha$, and the diagonal $AC = (BD =)d$ (i.e., the trapezoid is isosceles).
(d) if given its sides AD and BC, the angle between these sides, and the angle between the two diagonals.

5. Are there "Magical" Protractors? For the die-hards!

Yes, there are "magical" rulers: as we discovered earlier, you only need your straightedge and compass to do anything that a magical ruler would do. But are there "magical" protractors, i.e., tools that can divide any angle into as many equal parts as you wish?

We saw that we can bisect an angle into two equal angles. This means that you divide an angle into 4 parts, 8 parts, 16 parts, etc.: just keep subdividing into two equal parts until you get the desired 2^n equal sub-angles.

Question: Can you *trisect* an arbitrary angle into three equal sub-angles?

Let's check what Wikipedia says about this:[2]

> "**Angle trisection** is a classic problem of compass and straightedge constructions of ancient Greek mathematics. It concerns construction of an angle equal to one-third of a given arbitrary angle, using only two tools: an un-marked straightedge, and a compass.
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> With such tools, the task of **angle trisection is generally impossible**, as shown by Pierre Wantzel (1837). Wantzel's proof relies on ideas from the field of Galois theory – in particular, trisection of an angle corresponds to the solution of a certain cubic equation, which is not possible using the given tools. Note that the fact that there is no way to trisect an angle in general with just a compass and a straightedge does not mean that it is impossible to trisect all angles: for example, it is relatively straightforward to trisect a right angle (that is, to construct an angle of measure 30 degrees).
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> It is, however, possible to trisect an arbitrary angle, but using tools other than straightedge and compass. For example, *neusis construction*, also known to ancient Greeks, involves simultaneous sliding and rotation of a marked straightedge, which can not be achieved with the original tools. Other techniques were developed by mathematicians over centuries.
>
> Because it is defined in simple terms, but complex to prove unsolvable, the problem of angle trisection is a frequent subject of pseudomathematical attempts at solution by naive enthusiasts. The "solutions" often involve finding loopholes in the rules, or are simply incorrect."

Problem 7. (For the die-hards!)

(a) Investigate why trisecting angles is impossible. (Warning: to understand the full proof of impossibility may involve taking an ... upper-division college course in Galois Theory!)
(b) Investigate the neusis construction of trisecting an arbitrary angle.
(c) Investigate which angles CAN be trisected. In particular, see if you can understand:

Theorem 4. (Constructible Angles.) The angle θ may be trisected if and only the cubic polynomial $q(t) = 4t^3 - 3t - \cos(\theta)$ is reducible over the field extension $\mathbb{Q}(\cos(\theta))$.

(d) Investigate other methods of trisecting angles, e.g., via origami, an auxiliary curve, a marker ruler, a string, a "tomahawk," interconnected compasses, and others.

Thus, there are NO "magical" protractors, in the sense that we cannot duplicate with a straightedge and a compass the truly magical properties of a protractor that could divide angles into as many equal parts as we wished.

[2]Wikipedia: http://en.wikipedia.org/wiki/Angle_trisection