TILING PROBLEMS: FROM DOMINOES, CHECKERBOARDS, AND MAZES TO DISCRETE GEOMETRY

BERKELEY MATH CIRCLE

1. Looking for a number

Consider an 8×8 checkerboard (like the one used to play chess) and consider 32 dominoes that each may cover two adjacent squares (horizontally or vertically).

Question 1 (***). *What is the number N of ways in which you can cover the checkerboard with the dominoes?*

This question is difficult, but we will answer it at the end of the lecture, once we will have familiarized with tilings. But before we go on

Question 2. *Compute the number of domino tilings of a $n \times n$ checkerboard for n small. Try $n = 1, 2, 3, 4, 5$. Can you do $n = 6$...?*

Question 3. *Can you give lower and upper bounds on the number N? Guess an estimate of the order of N.*

2. Other settings

We can tile other regions, replacing the $n \times n$ checkerboard by a $n \times m$ rectangle, or another polyomino (connected set of squares). We say that a region is *tileable* by dominos if we can cover entirely the region with dominoes, without overlap.

Question 4. *Are all polyominoes tileable by dominoes?*

Question 5. *Can you find necessary conditions that polyominoes have to satisfy in order to be tileable by dominos?*

What if we change the tiles? Try the following questions.

Question 6. *Can you tile a 8×8 checkerboard from which a square has been removed with triminos 1×3 (horizontal or vertical)?*

Question 7 (*). *If you answered yes to the previous question, can you tell what are the only possible squares that can be removed so that the corresponding board is tileable?*

Date: February 14th 2012; *Instructor:* Adrien Kassel (MSRI); for any questions, email me.

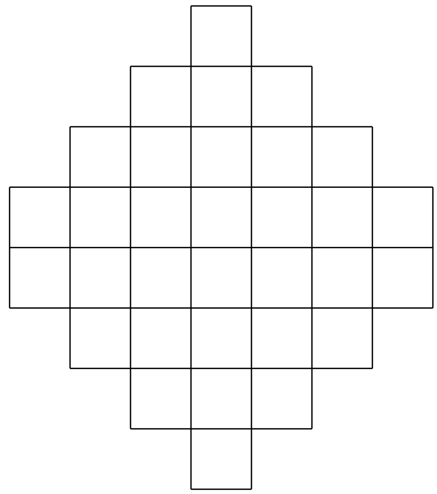

FIGURE 1

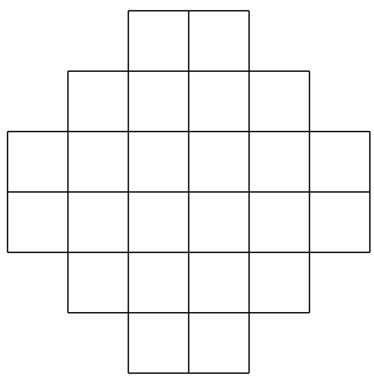

FIGURE 2

Now, what about other grids?

Question 8. *Can you tile a large hexagon with lozenges?*

Consider the following shape, Figure 1.

Question 9. *How many possibilities are there?*

And this one, Figure 2.

Question 10. *How many tilings for Figure 2?*

This is an example of how the *boundary conditions* (shape of the border of the domain) influence the number of possible tilings.

3. An exact count

Question 11. *What is the number of domino tilings of a $2 \times n$ rectangle?*

4. Graphs

Recall that a *graph* $\mathcal{G} = (V, E)$ is a set of *vertices* along with a set of *edges* that each link two vertices. A graph is *connected* if all pair of vertices can be joined by a path of edges.

A *perfect matching* is a subset of the edges $m \subset E$ such that each vertex is the end vertex of exactly one edge $e \in m$.

Question 12. *Show that a domino tiling of the checkerboard corresponds bijectively to a perfect matching of a well-chosen graph.*

Question 13. *Can you find necessary conditions on graphs to have perfect matchings?*

A graph whose vertices can be colored in black and white such that black and white alternate is called *bipartite*.

Question 14. *Give a necessary condition on a bipartite graph to have perfect matchings.*

Question 15. *Show that a graph is bipartite iff all its cycles are of even length.*

Not all graphs can be drawn in the plane or (equivalently) on a sphere. Those that can be drawn in the plane (or equivalently on the sphere) are called *planar*.

Question 16. *Can you give an example of a non-planar graph?*

Question 17 (*)**.** *Can you show that all finite graphs can be drawn in $\mathbf{R}^3$?*

A *tree* is a graph without cycle.

Question 18. *Show that a tree is a connected graph with exactly as many edges as the number of vertices minus* 1*.*

A *spanning tree* of a connected graph is a subgraph which is a tree and connects all vertices.

Question 19. *Can you show that all finite graphs have spanning trees?*

5. Euler characteristic

Question 20. *Show that for any connected planar graph $\mathcal{G} = (V, E)$,*

$$|V| - |E| + |F| = 2\,,$$

where F is the set of faces (including the unbounded one).

Question 21. *Use Euler's formula to prove rigorously that a pentagon with all of its diagonals is non planar.*

A *regular polyhedron* is a convex glueing of polygons, such that all faces are the same polygon (with p edges), and at each vertex there is the same number q of polygons glued. Schläfli symbol: $\{p, q\}$. For example the cube or the tetrahedron are regular polyhedra.

Question 22. *Show that there are at most* 5 *regular polytopes.*

Question 23. *In fact, there are exactly* 5 *since you can probably name* 5 *of them : the cube, the tetrahedron, ... ?*

As you may know, Euler first started off graph theory to solve the problem of the seven bridges of Königsberg.

Question 24 (*). *Have you heard of this problem? Can you solve it?*

6. Height function

We are going to associate an integer-valued function (stepped surface) over the vertices of the squares of a tileable polyomino (for example a $2n \times 2n$ chessboard).

Here is the rule. Start with a $2n \times 2n$ chessboard and a tiling T. We are going to define a height function h_T which is an integer valued function over the vertices of all squares. Set h_T to be 0 at the lower left corner v_0. Now follow the rule: for any vertex v, choose a path from the corner v_0 to v following the sides of the dominoes, and apply the following rule: Look at the square on your right

- if it is black, increase the height by 1,
- if it is white, decrease the height by 1.

If there is no square on your right, look at the one on your left and if it is black, decrease the height by 1, and if it is white, increase by 1.

Question 25. *Show that this height is well-defined, i.e. show that it doesn't depend on the choices of paths you make.*

Notice that the values of the height function on the boundary of the domain do not depend on the tiling.

Actually, height functions enable to rephrase the problem of tileability of a polyomino in terms of a boundary value problem for a function over the vertices of the polyomino. There exists a tiling iff there exists a stepped surface (which satisfies certain constraints) with boundary values given by the shape of the polyomino.

What happens on other shapes?

Question 26. *Can you define a height function for lozenge tilings of a hexagon in the triangular grid?*

7. Temperley's bijection

We are now going to associate to any tiling of a checkerboard, a spanning tree of a certain graph.

Temperley introduced the following construction. Start with a $(2n+1)\times(2n+1)$ chessboard. We suppose that the chessboard has black squares at all its 4 corners. Remove the upper-right corner. Call it the *root*. Let $\mathcal{G}$ be the subgraph consisting of the square lattice of black squares at distance 2 appart only and containing the corners. We are going to define a tree t_T on the subgraph $\mathcal{G}$ and rooted at the root. Start at any black square in $\mathcal{G}$ and draw a directed edge to the only direction given by the domino that covers it. Continue until you reach the removed corner. Continue this procedure until every black square is connected to the root.

Now draw lines between the neighboring squares that remain without crossing the tree t_T. This yields a family of trees all rooted on the exterior face of the chessboard.

Question 27. *Show that Temperley's correspondence is a well-defined bijection between the set of domino tilings of the checkerboard and the set of trees on $\mathcal{G}$.*

What about other shapes?

Question 28 (***)**.** *Can you define a spanning tree that encodes all information in the case of lozenge tilings of a hexagon in the triangular grid?*

8. Ordering tilings

Take the hexagon and a lozenge tiling of it.

Question 29. *Can you find a way to order the set of all tilings, that is find a way to compare them, say which out of two is "bigger" than the other one?*

9. Permutations and determinant

A *permutation* over n elements is a bijection of the set $\{1,\cdots,n\}$, that is a shuffling of these n elements. We call $\mathcal{S}_n$ the set of permutations.

Question 30. *Show that any permutation can be written as a product of cycles with disjoint supports.*

For any cycle c, define $\varepsilon(c)=(-1)^{|c|-1}$, where $|c|$ is the length of the cycle. For a permutation $\sigma=\prod_i c_i$, define $\varepsilon(\sigma)=\prod_i \varepsilon(c_i)\in\{-1,1\}$. This is called the *signature* of the permutation.

Recall that a *square matrix* of size n is a square array of numbers of size $n\times n$. Now for any matrix $M=(M_{i,j})$, define its determinant to be

$$\det M=\sum_{\sigma\in\mathcal{S}_n}\varepsilon(\sigma)\prod_i M_{i,\sigma(i)}\,.$$

Define the *Kasteleyn matrix* to be a square matrix with rows indexed by black vertices and columns by white vertices. Define $K_{b,w} = 0$ if b is not a neighbor of w, and $i = \sqrt{-1}$ or 1 according to the fact that bw is horizontal or vertical.

Question 31. *Show that* $\det K = \sum_T z_T$*, where the sum is over all possible tilings, and for any tiling* T*, we have* $z_T = 1, -1, i,$ *or* $-i$*.*

Question 32 (*)**.** *Can you show that in fact* $|\det K| = |\{tilings\}|$ *is the number of tilings?*

Now, using some linear algebra, we can compute determinants. We will show in class how to compute this one.

Question 33 (**)**.** *Find the eigenvalues of* K*. Conclude that* $N =?$.

10. Applications, current research, and open problems

Now enjoy the slides.

10.1. **The effect of boundary conditions in the infinite volume limit.** The arctic circle phenomena for domino tilings of the Aztec diamond. See slides.

10.2. **Random fractals.** We add some randomness in the picture. We consider a large $n \times n$ chessboard, and a uniformly chosen random domino tiling of it. This yields a random discrete surface h_T and a random tree t_T. See slides.

11. Bibliography

- Leonardo Pisano Bigollo (c. 1170 – c. 1250) also known as Leonardo of Pisa, Leonardo Pisano, Leonardo Bonacci, Leonardo Fibonacci, was an Italian mathematician.
- Leohnard Euler (15 April 1707 – 18 September 1783) was a Swiss mathematician and physicist.
- Pieter Willem Kasteleyn (October 12, 1924 – January 16, 1996) was a Dutch physicist.
- Harold Neville Vazeille Temperley (born 4 March 1915) is a British mathematical physicist.
- Michael Ellis Fisher (born 3 September 1931 in Trinidad, West Indies) is a mathematical physicist.