Taking Graphs Deeper

The complete graph on 5 vertices is denoted K_5. It is constructed by drawing 5 dots, and then connecting each dot to every other dot with an edge.

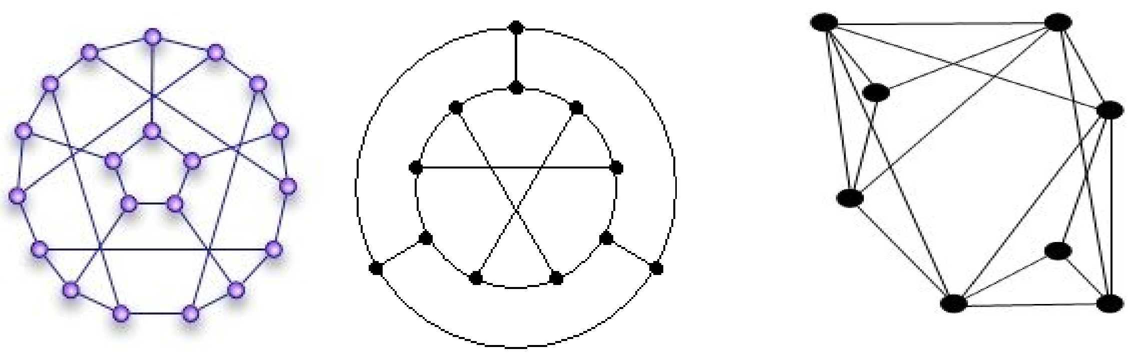
1. Try to draw this so that no two edges cross.

When it is possible to draw a graph so that no two edges cross, the graph is called *planar*. You can probably now guess what a complete graph on *n* vertices would be. A similar type of graph is called the complete *(3,3)* bipartite graph. It is denoted $K_{3,3}$. To construct it, draw 3 red dots and 3 blue dots, and then connect each red dot to each blue dot with an edge.

2. Try to draw $K_{3,3}$ so that no two edges cross.

You may have guessed that K_5 and $K_{3,3}$ are not planar. This is correct.

3. For each of the following graphs, find a planar embedding or a contraction of a subgraph that is isomorphic to K_5 or $K_{3,3}$ (The fact that this is possible is known as Kuratowski's theorem.)



http://scientopia.org/blogs/goodmath/2007/07/02/coloring-planar-graphs/
http://people.qc.cuny.edu/faculty/christopher.hanusa/courses/Pages/bu/381sp08/homework.html

4. If a graph has a non-planar subgraph, then it is non-planar. Show that a graph with a non-planar contraction is non-planar. Equivalently, a subgraph of a planar graph is planar, and the contraction of a planar graph is planar.
5. Embed each of the above graphs in a surface of minimal genus.

If a property of graphs is preserved by subgraphs and contractions, then graphs that do not satisfy the property can be characterized in a nice way. Namely given any graph that does not satisfy the property, one can first take a minimal subgraph that still does not satisfy the property, and then contract it as much as possible so that the result still does not satisfy the property. The result will be called *minor minimal*. There is always a finite collection of minor minimal graphs for a property satisfying these two conditions. For example the minor minimal non-planar graphs are K_5 and $K_{3,3}$.

Open Problem: Describe the collection of all minor minimal non-genus 1 graphs. (There are more than 800.)

The fact that there are so many, makes me think that we are using the wrong reduction. It is natural to consider subgraphs and contractions, but there is probably some other simplifying operation(s).