Along the Euler Line

Berkeley Math Circle – Intermediate

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Note: We shall work on the problems below over the next two circle sessions of BMC-intermediate. Bring this handout with you to both sessions. Try to understand what the problems say and draw pictures for them as best as you can. You are not expected to be able to solve the problems on your own, at least not in the beginning of this Geometry module at BMC–intermediate. Bring drawing tools to the sessions too: compass, ruler, triangles, protractor, pencils, erasers.

You should download the free geometer software http://geometer.org/geometer/index.html from Tom Davis's web page, in order to be able to reconstruct and dynamically change the constructions in our sessions.

1. A Bit of Related History (adapted from *Wikipedia*)

Leonhard Euler (15 April 1707 – 18 September 1783) was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function: he was the first to write $f(x)$ to denote the function f applied to the argument x. He also introduced the modern notation for the trigonometric functions, the letter e for the base of the natural logarithm (now also known as *Euler's number*), the Greek letter Σ for summations and the letter i to denote the imaginary unit. The use of the Greek letter π to denote the ratio of a circle's circumference to its diameter was also popularized by Euler, although it did not originate with him. He is also renowned for his work in mechanics, fluid dynamics, optics, and astronomy.

Euler spent most of his adult life in St. Petersburg, Russia, and in Berlin, Prussia. He is considered to be the preeminent mathematician of the 18th century, and one of the greatest of all time. He is also one of the most prolific mathematicians ever; his collected works fill 6080 quarto volumes. A statement attributed to Pierre-Simon Laplace expresses Euler's influence on mathematics: *"Read Euler, read Euler, he is our teacher in all things,"* which has also been translated as *"Read Euler, read Euler, he is the master of us all."*

Euler was featured on the sixth series of the Swiss 10-franc banknote and on numerous Swiss, German, and Russian postage stamps. The asteroid 2002 Euler was named in his honor.

2. What is Euler Line?

Definition 1. *Euler line* is the line that passes through the following three distinguished points in a triangle: the orthocenter, the circumcenter, and the centroid of the triangle.

This definition is more trouble than help: it poses more questions than it gives answers to! Indeed, what are the *orthocenter*, the *circumcenter*, and the *centroid* of a triangle, and why should they be *collinear*, i.e., lying on the same line? Here are some answers:

Definition 2. The intersection of the three *altitudes* in a triangle is called the *orthocenter* of the triangle. The center of the circle circumscribed about a triangle is called the *circumcenter* of the triangle. The intersection of the three *medians* in a triangle is called the *centroid* (or the *medicenter*) of the triangle.

Perhaps, some of the notions in Definition 2 are familiar to you, and, perhaps, the above is of some help in deciphering the meaning of the *Euler line*. But it is obvious that Definition 2 still continues to pose more and more questions to us, forcing us descend further down to the foundations of *Plane Geometry*. For example, why should the three *altitudes* in a triangle intersect in a single point? For three lines to intersect in a single point, the lines must be in a very special position to each other; we call such lines *concurrent*. Similarly, why should the three *medians* also be concurrent? Or why should every triangle have a *circumcircle* and how can we easily find the centercenter?

2.1. **The Circumcenter of a Triangle.** Well, at least those who attended BMC-beginners last spring should remember the answer to the last question, which is in the form of a *theorem* (i.e, a *proven* statement).

Theorem 1 (Circumcenter)**.** *For every triangle there is exactly one circle (called the* circumcircle*) that passes through the three vertices of the triangle. Moreover, the center of the circumcircle is the intersection of the three* perpendicular bisectors *of the sides of the triangle; this center is called the* circumcenter *of the triangle.*

Again, we might recall from last year that

Definition 3. The *perpendicular bisector* of a segment AB is a line l which is both perpendicular to AB and passes through the midpoint of AB.

Theorem 2 (Perpendicular Bisector)**.** *The perpendicular bisector of a segment AB is the locus of all points in the plane that are* equidistant *from A and B, i.e., all points X such that $|XA| = |XB|$.*

Here are a couple of warm-up exercises for you:

Exercise 1. Using only a compass and a straightedge (and a pencil, of course), but without using the markings on your ruler (i.e., no measuring is allowed!):

(a) draw a segment AB;
(b) find a point X that is equidistant from A and B, i.e., $|XA| = |XB|$;
(c) find another point Y that is equidistant from A and B;
(d) connect X and Y to form a line; mark the intersection point of line XY and AB by M;
(e) prove that M is the midpoint of AB;
(f) prove that line XY is perpendicular to AB. Is line XY the *perpendicular bisector* of AB?
(g) select a third point Z on line XY and prove that Z is also equidistant from A and B.

Exercise 2. Given $\triangle ABC$, let l, m, and n be the perpendicular bisectors of its three sides.

(a) If l and m intersect in a point O, prove that O is equidistant from all vertices of the triangle, i.e., $|OA| = |OB| = |OC|$.
(b) Is O the circumcenter of $\triangle ABC$? Why?
(c) Why should the third line n also pass through O?

Exercise 3. Using only a compass and a straightedge, find the circumcenter and circumcircle of

(a) an acute triangle (all of its three angles are acute);
(b) an obtuse triangle (one of its angles is obtuse);
(c) a right triangle (one of its angles is right).

What do you notice about the position of the circumcenter with respect to the triangle in each of these three cases? Can you prove your conjecture? (This will require knowing some circle geometry from last spring sessions at BMC.)

Theorem 3 (Location of Circumcenter)**.** *The circumcenter of an acute triangle lies inside the triangle; for an obtuse triangle its circumcenter lies outside the triangle; while in the case of a right triangle its circumcenter is the midpoint of the hypotenuse.*

2.2. **The Orthocenter of a Triangle.** Now that we know how to draw perpendicular bisectors to segments, can we extend this knowledge to constructing any perpendiculars without using the right angle of your triangle/ruler? Why would we be interested in this? Because the next distinguished point in any triangle, its *orthocenter*, is the intersection of the *altitudes* of that triangle.

Definition 4. An *altitude* of a triangle is a segment that passes through a vertex of the triangle and is perpendicular to the opposite side of the triangle. The point where the altitude lands on the opposite side (making a right angle with it) is called the *foot* of the altitude.

The following questions will make you more comfortable with altitudes and orthocenters:

Exercise 4a. Using only a compass and a straightedge:

(a) draw a segment AB;
(b) select a point X on the line AB;
(c) erect a perpendicular to AB through X, i.e., a line XY that is perpendicular to AB.

Exercise 4b. Using only a compass and a straightedge:

(a) draw a segment AB;
(b) select a point X off the line AB;
(c) drop a perpendicular from X to AB, i.e., a line XY that is perpendicular to AB.

Compare the solutions to Exercises 4a and 4b. From them, we can conclude that we don't need anymore the right angle of our drawing tools: whatever we can do with such a right angle, we can also do the same thing with only a compass and a straightedge. Why? Explain. Can we push this further to drawing *parallel* lines?

Exercise 4c. Using only a compass and a straightedge:

(a) draw line AB;
(b) select a point X off the line AB;
(c) draw a line through X that is parallel to AB.

In part (c), did you first drop a perpendicular from X to AB, denoting the foot of the perpendicular by Y, and then erect a perpendicular through X to line XY? This should have resulted in the desired parallel line to AB. Now let's go back to altitudes and orthocenters.

Exercise 5. Using only a compass and a straightedge, explain how you can draw the three altitudes AA_1, BB_1 and CC_1 of $\triangle ABC$. Now investigate if these three altitudes are always concurrent for

(a) an acute triangle; (b) an obtuse triangle; (c) a right triangle.

What do you notice about the position of the *orthocenter* (the intersection of the three altitudes) with respect to the triangle in each of these three cases? Can you prove your conjecture?

Theorem 4 (Orthocenter)**.** *In any triangle, the three altitudes are concurrent. Moreover, their common intersection, the orthocenter, lies inside acute triangles, outside obtuse triangles, and coincides with the vertex of right angle in a right triangle.*

The full proof of this theorem may involve more techniques than the scope of this geometry module. However, you can think of the above theorem as a long-term project.

2.3. **The Centroid of a Triangle.** We finally get to the third distinguished point in every triangle: its *centroid*, or the intersection of the three *medians* in a triangle.

Definition 5. A *median* of a triangle is a segment connecting a vertex of the triangle with the midpoint of the opposite side of the triangle.

It is a good thing we know how to find midpoints of segments using only a straightedge and a ruler!

Exercise 6. Using only a compass and a straightedge:

(a) explain how you can draw the three medians AA_2, BB_2 and CC_2 of $\triangle ABC$;
(b) investigate if these three medians are always concurrent;
(c) where is the centroid with respect to the triangle for an acute triangle; an obtuse triangle; or a right triangle? Is there anything special about each case and the centroid?
(d) notice the position of the centroid on each of the three medians? Can we precisely locate the centroid by drawing only *one* of the medians? Why?
(e) given segment DE, divide it into 3 equal parts (no rulers!). Why am I asking this?

Theorem 5 (Centroid)**.** *In any triangle, the three medians are concurrent. Moreover, the centroid divides each median in ratio* 2 *to* 1*, counted from the vertex of the triangle.*

This is a theorem that you can attempt to prove, using only criteria about similar triangles. It's going to be hard if you are doing geometry and proofs like this for the first time, but it will be worth it! And while you are at it, prove a related theorem about *midsegments* of triangles:

Theorem 6 (Midsegment)**.** *The segment connecting the midpoints of two sides in a triangle is called a* midsegment *of the triangle. Every midsegment is parallel to and half of the size of the third side of the triangle. Further, the three midsegments in a triangle form a smaller triangle similar to the original and of 4 times smaller area.*

And now, here is a nice little application of the Midsegment Theorem:

Exercise 7. Can you use the properties of a midsegment to devise a different way of drawing parallel lines? Thus, can you do Exercise 4c(c) by only finding one midpoint of a certain segment and making your parallel line go through a midsegment of a certain triangle? Try it!

By the way, why is the centroid of a triangle called a *centroid*? Have you heard of the expression "*center of mass of* $\triangle ABC$"? Very likely you have! If you hang your triangle (conveniently made of cardboard!) on a string from the centroid M, you will discover that the triangle stays horizontal to the floor! This is why point M is called the *centroid* (or *center of mass*) of $\triangle ABC$. Try it!

2.4. **Putting Everything Together.** The Euler line combines the three distinguished points of a triangle in one elegant statement. Let's experiment first:

Exercise 8. Draw $\triangle ABC$, its orthocenter H, circumcenter O, and centroid M. What do you notice about the relative positions of H, O, and M? Does your conjecture still hold true when you change the triangle? Does it depend on whether the triangle is acute, obtuse, or right?

Theorem 7 (Euler, 1765)**.** *The orthocenter H, circumcenter O, and centroid M of a triangle are always collinear, with point M between H and O, twice as close to O as to H, i.e.,* $|MH| = 2|MO|$. *The line OMH is called* Euler's line.

The proof of Euler Line Theorem is actually not nearly as hard as the statement of the theorem may sound; indeed, the proof involves only one pair of similar triangles and an application of a theorem done earlier in this very handout (which theorem?). However, a warning bell rings: is there always an Euler *line*? What if the 3 distinguished points actually collide on top of each other?

Exercise 9. When do the orthocenter H, circumcenter O, and centroid M coincide?

(a) Draw an equilateral triangle. What can you say about our three distinguished points? Is there an Euler *line*? Prove your answers.
(b) Show that if two of our three distinguished points coincide, then the triangle is equilateral, and show that the third point also coincides with the other two.
(c) Conclude that the Euler line exists as a *line* exactly for non-equilateral triangles.

3. What Else is There Along the Euler Line?

As it turns out, there are a number of other distinguished points on the Euler line. But we need to define them first, before getting unnecessarily scared by their fancy names.

Definition 6. The *nine-point circle*[1] of a triangle is the circle passing through the following 9 points: the 3 midpoints of the sides of the triangle, the 3 feet of the altitudes, and the 3 halfway points between the orthocenter and the vertices of the triangle. The center of this circle is called the *nine-point center*, often denoted by N.

Surprise, surprise! The nine-point center N not only happens to lie on the Euler line, but in a very distinguished place along it:

Theorem 8 (Nine-Point Circle)**.** *The 9 points described above indeed lie on a circle (we say that they are* concyclic*) and the center N of this circle is halfway between the orthocenter H and the circumcenter O of the triangle. Hence, the nine-point center N also lies on the Euler line.*

So far, we talked about altitudes, medians, and perpendicular bisectors; but what happened to the *angle bisectors* in a triangle?

Definition 7. The *incenter* I of a triangle is the center of its *inscribed* circle, i.e., the circle that is tangent to all three sides of the triangle.

Theorem 9 (Incenter)**.** *The three angle bisectors in a triangle are concurrent; moreover, they intersect in the incenter I of the triangle.*

Exercise 10. Can you guess if the incenter I lies on the Euler line? Determine where I is with respect to the Euler line, for the following triangles:

(a) an equilateral triangle? (b) a right triangle? (c) a scalene triangle? (d) an isosceles triangle?

Theorem 10 (Incenter and Euler Line)**.** *The incenter of a triangle lies on the Euler line exactly when the triangle is isosceles. In such a case, the Euler line is the altitude (also simultaneously, median, perpendicular bisector, and angle bisector) towards the base of the isosceles triangle.*

4. Some Further Challenges for the Die-Hards Only!

Problem 1 (Altitudes and Circumcircle)**.** *$\triangle ABC$ is inscribed in circle k. The extensions of the two altitudes AE and BD of $\triangle ABC$ intersect k in points A_1 and B_1, respectively. If $\angle C = 60°$, prove that $AA_1 = BB_1$. Is the converse true?*

Problem 2 (BAMO 2005)**.** *If two of the medians are equal then the triangle is isosceles.*[2]

Problem 3 (BAMO 2006)**.** *In $\triangle ABC$, three points A_1, B_1 and C_1 are selected on sides BC, CA, and AC, respectively, so that the segments AA_1, BB_1 and CC_1 intersect in some point P. Prove that P is the centroid of $\triangle ABC$ if and only if P is the centroid of $\triangle A_1B_1C_1$.*

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[1] "The nine-point circle is also known as Feuerbach's circle, Euler's circle, Terquem's circle, the six-points circle, the twelve-points circle, the n-point circle, the medioscribed circle, the mid circle, or the circum-midcircle." [Wikipedia]

[2] BAMO is the Bay Area Mathematical Olympiad. We will proctor the exam for all BMC-Upper students at the end of February 2012.

5. Necessary Tools in Every Young Geometer's "Toolbox"

Congruence and *Similarity Criteria for Triangles* are fundamental tools in geometry. Below is a list of all such criteria that may be useful in any geometry sessions you will encounter. For each criteria, draw a relevant picture of two triangles that are similar, label their vertices, mark the angles in your triangles that are supposed to be equal and the sides that are supposed to enter in equal ratios, and write down (in letter notation) the conditions that are satisfied by the criteria. You are *not* supposed to prove these criteria.

5.1. Criteria for Congruence of Triangles.

Theorem 11 (Congruence). *Two triangles are* congruent *if one of the following criteria is satisfied:*

(a) *(SAS) Two sides and an angle between them in one triangle are correspondingly equal to two sides and an angle between them in another triangle.*
(b) *(ASA) Two angles and the common side that they share in one triangle are correspondingly equal to two angles and the common side that they share in another triangle.*
(c) *(SSS) The three sides in one triangle are correspondingly equal to the three sides in another triangle.*
(d) *(ASS) Two sides and an angle opposite to one of these sides in one triangle are correspondingly equal to two sides and an angle opposite to one of these sides in another triangle. In addition, the side opposite the angle is the largest side in each triangle. (*Note: *In $\triangle ABC$, side BC lies opposite to $\angle BAC$.)*
(e) *(ASS-corollary) The hypotenuse and one of the legs in one* right *triangle are correspondingly equal to the hypotenuse and one of the legs in another* right *triangle. (Note: This criteria is a special case of the (ASS) criteria in (d). Why?)*

5.2. Criteria for Similarity of Triangles.

Although two congruent triangles are also similar, being congruent is a very special case of being similar. Thus, make sure that the triangles in your pictures are **not** congruent but are just **similar**.

Theorem 12 (Similarity). *Two triangles are* similar *if one of the following criteria is satisfied:*

(a) *(RAR) For two sides and an angle between them in one triangle and analogous three elements in the other triangle, the two angles are equal, and the two ratios of the corresponding sides in the two triangles are equal.*
(b) *(AA) Two angles in one triangle are correspondingly equal to two angles in another triangle. (What happens with the third angles in the triangles?)*
(c) *(RRR) The three ratios of the sides in one triangle to the corresponding sides in another triangle are all equal.*

5.3. Necessary Basic Geometry Theorems and concepts.

In order to solve some of the problems in this handout, you will need the following two well-known theorems:

Theorem 13 (Parallelogram). *The two diagonals in a parallelogram bisect each other.*

Theorem 14 (Trapezoid). *A trapezoid is isosceles iff its two diagonals are equal in length.*

Definition 8. A *dilation* in the plane is a transformation of the plane that has for its *center* some point O and for its *ratio* some number r, where r could be positive or negative. Given a point X in the plane, send X to point X_1 on line OX so that $|OX_1| = |r| \cdot |OX|$; more precisely, if $r > 0$, then X_1 lies on *ray* $\overrightarrow{OX}$; but if $r < 0$ then X_1 lies on the ray opposite to *ray* $\overrightarrow{OX}$.