Walking around infinite graphs.

The main topics of tonight's talk are random walks and ponzi schemes. The following results are some sort of background to the talk. (But we are unlikely to get to everything.)

We will apply these ideas to understand random walk on the integers, on higher dimensional lattices, on some trees and also the possibility of making successful ponzi schemes on various graphs.

I. Review of Basic Probability.

A random variable is a function that depends on a random occurrence. For instance it can be the number of times a coin flipped 10 times came out heads. This will have 11 possible values as its outcome.

The *expectation value* of a random variable is its weighted average, where we weight an event according to its likelihood. It is $E(f) = \Sigma\, np(f(x) = n)$, where $p(f(x) = n)$ means the probability that $f(x) = n$.

So the expectation of the function mentioned above is 5.

One way to see this is because $E(f+g) = E(f) + E(g)$.

Two events are *independent* if the probability of both occurring is the product of the probability of each individually. (If there are several events, we suppose that this is true for all collections of the events.)

II. Stirling's approximation.

For large n, there is a very useful approximation to n!

$$n! \sim \sqrt{(2\pi n)}\, n^n e^{-n}$$

where e is a constant (2.7182818….).

III. Max Flow Min Cut Theorem.

The maximum flow from a source to a sink in a graph is determined by the smallest cut separating them.

IV. Hall's Marriage Theorem.

Suppose that two sets are given, A and B, and we have a function from $F: A \rightarrow$ Subsets of B (each $a \in A$ announces – via the function F -- which elements of B, she'd be happy with). The goal is to find a function $f: A \to B$, so that for every a, $f(a) \in F(a)$, and if $a \neq a'$, $f(a) \neq f(a')$.

Hall's theorem says that this is possible if and only if for every subset S of A, $\#F(S) \geq \#S$.

Problems based on the talk. (They are not easy. I don't know exact answers to all of them.)

1. a. For a random permutation of $\{1,2,\ldots n\}$, how likely is it to have a cycle that contains at least half of the integers?
   b. Show that the expected number of cycles of odd length is greater than the expected number of even length, but by less than one!

2. A bacterium has a probability of 1/3 of dying before undergoing mitosis and splitting into two identical bacteria. How likely is it that this line of bacteria will be immortal (ie. that no generation dies before reproducing)? What if the probability were ½? What if had a probability ½ of dying, but if it survived, it would split into 3 cells?

3. Suppose that one has a 100 x 100 x 100 cubical building, where each 1 x1 x 1 produces 1 liter of waste each day, around how much capacity must some pipe have to carry the waste out of the building? (Estimate it!) What if the building were 1000 x 1000 x 1000? Why does this make the Star Trek Episode where there was a cell as large as a Galaxy seem unlikely?

4. Which of the following graphs have positive non-constant harmonic functions? (Recall that a function on a graph is harmonic if the value at every vertex is the average among the values of its neighbors.)
(a). Z, (b). (this one is very hard; I don't think you know enough, but you can try) Z^2, (c). The 3-valent tree, (d). (this one is tricky, but I think you can do it!) A tree that alternates between being 3 valent and 2 valent, where the n-th generation is 3-valent exactly if n is a power of 2.

5. On the 3-valent tree, describe an escape plan so that each vertex is assigned a path "to infinity", and no vertex lies on more than two such paths.

6. Prove the following corollary to Hall's theorem.

Theorem: Assume that we are in the situation of Hall's theorem, then there are two functions $f,g: A \to B$, so that for all a, a', $f(a) \neq g(a)$, and $\{f(a), g(a)\} \cap \{f(a'), g(a')\} = \varnothing$ if and only if for all subsets S of A, $\#F(S) \geq 2\#S$.

Key words for further study: *Graph, Random Walk, Markov Chain, Polya's theorem, Hitting time, Branching process, Harmonic Functions, Poisson formula, Hall's Marriage* *lemma, Amenable group, Linear Programming (duality), Max Flow Min Cut – the Ford-Fulkerson theorem.*