Euler's Magic Homework

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During the class I will use the trigonometric formulae

$$\begin{aligned}\cos(a+b) &= \cos a\cos b - \sin a\sin b\\ \cos(a-b) &= \cos a\cos b + \sin a\sin b\\ \cos a &= \sin(\pi/2 - a).\end{aligned}$$

1. If you don't already know the answer, find the value of the infinite sum

$$1 + r + r^2 + r^3 + \cdots$$

for each number r satisfying $-1 < r < 1$. (Hint: you want to find a formula for the partial sum $1 + r + r^2 + \cdots + r^n$: try multiplying the sum by $1 - r$.)

2. If you know some calculus, draw a picture to explain why

$$\frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < \int_1^n \frac{1}{x^2}\,dx$$

and use this to give a different proof of the estimate in class.

3. Find a formula for the sum

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{(n-1)n}$$

and give yet another proof of the estimate.

4. Recall that the 5^{th} Chebyschev polynomial $T_5(x) = 16x^5 - 20x^3 + 5x$. Express $T_5(x) + 1$ in the form

$$T_5(x) + 1 = (x+1)(ax^2 + bx + c)^2$$

where a, b and c are whole numbers. Find where $T_5(x) = -1$ in two different ways: by solving the polynomial equation and by using the fact that $T_5(\cos\theta) = \cos 5\theta$. Find an exact expression for $\cos\frac{\pi}{5}$.

5. If you have some mathematical software write a programme to calculate the Chebyschev polynomials of larger degree and plot their graphs. Compare the graph of $y = T_n(x)$ with that of $y = \sin nx$ for odd numbers n.