Euler's Magic Series

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November 19, 2011

1 Infinite series

Infinite sums play an essential role in much of mathematics. Some of the most important functions are given by infinite sums: for example the exponential function:

$$e^x = 1 + x + x^2/2 + x^3/6 + \cdots + x^n/n! + \cdots$$

We cannot make sense of infinite sums just by knowing about finite sums: we need to decide what it means to add up infinitely many numbers. The definition that has been found to work best is the following. Given an infinite sum

$$a_1 + a_2 + a_3 + \cdots$$

we look at the sequence of what are called *partial sums* obtained by adding the terms one after another

$$\begin{aligned} &a_1 \\ &a_1 + a_2 \\ &a_1 + a_2 + a_3 \\ &\vdots \end{aligned}$$

If these partial sums approach some fixed number A as we keep adding, then we say that the infinite sum *converges* and that the sum is A.

For example if the terms are $1/2, 1/4, 1/8, \ldots$ ther partial sums are

$$\begin{aligned} 1/2 &= 1/2 \\ 1/2 + 1/4 &= 3/4 \\ 1/2 + 1/4 + 1/8 &= 7/8 \\ &\vdots \end{aligned}$$

and it is easy to see that the numbers approach 1. So we say that

$$1/2 + 1/4 + 1/8 + 1/16 + \cdots = 1.$$

There are very few natural looking series for which we can calculate the sum exactly. We usually need to use other methods to show that the sums makes sense.

2 Euler's series

Bernoulli studied several infinite series and demonstrated among other things the convergence of the sum

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

whose terms are the reciprocals of the perfect squares. But he was unable to determine the value of the sum. Let us calculate some partial sums and try to see the pattern.

$$\begin{array}{lcl} 1 & = & 1 \\ 1 + 1/4 & = & 5/4 \\ 1 + 1/4 + 1/9 & = & 49/36 \\ 1 + 1/4 + 1/9 + 1/16 & = & 205/144 \\ \vdots & & \end{array}$$

It is not easy to see a pattern and things get worse if we continue. It is highly unlikely that there is any pattern that can be easily described: we shall see some evidence for this later. How then can we show that the sum converges?

The simplest thing one can say about the partial sums is that they get larger because we are adding a positive quantity each time. An increasing sequence can do one of two things. It could increase without bound, eventually surpassing any number we can write down. But if it is bounded; if it is trapped below some ceiling; then it is forced to approach some number. It might not necessarily converge to the ceiling we identified, but it will converge to *something*.

So if we can show that for each n,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} \leq 2$$

then we will know that the sum converges without knowing what it converges to (although we will know that the sum is less than 2). I shall use mathematical induction to show that, for each n,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

which is a bit stronger than what we want. At the moment you may not be able to see why I chose $2 - \frac{1}{n}$ (or even the ceiling 2): once you see the argument you will have a better chance to understand why.

Observe that if $n = 1$ then both sides of the inequality are equal to 1. We shall show that if the inequality holds for any given n then it also holds for the next number $n+1$. Since it holds for $n = 1$ this ensures that it holds for $n = 2$ and then $n = 3$ and so on. Now, if we know the inequality for n we have that the sum of $n+1$ terms satisfies

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}.$$

So it will be enough to show that

$$2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}.$$

This is the same as

$$\frac{1}{(n+1)^2} \leq \frac{1}{n} - \frac{1}{n+1}$$

and the right side is $\frac{1}{n(n+1)}$ which is indeed larger than the left side.

This argument shows that the infinite sum converges to something but it doesn't tell us what the sum is (merely that it's less than 2). In about 1735 Euler stunned the mathematical world by showing that

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

which helps to explain why you can't find a pattern to the partial sums.

Euler's discovery is shocking but his argument was even more so. He began like this. Consider the function $x \mapsto \sin x$. If you have seen some calculus you probably know that the sine function can be written as an infinite sum

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \tag{1}$$

and so it looks a bit like a polynomial. If you haven't seen calculus I will ask you to accept this but here is a graph of $y = \sin x$ and of $y = x - x^3/6 + x^5/120$ showing how similar they are.

There is a fundamental property of polynomials (which is not possessed by other functions): namely, that if you know where a polynomial is equal to zero, you can write the polynomial as a product of factors. If p is a polynomial of degree 3 which

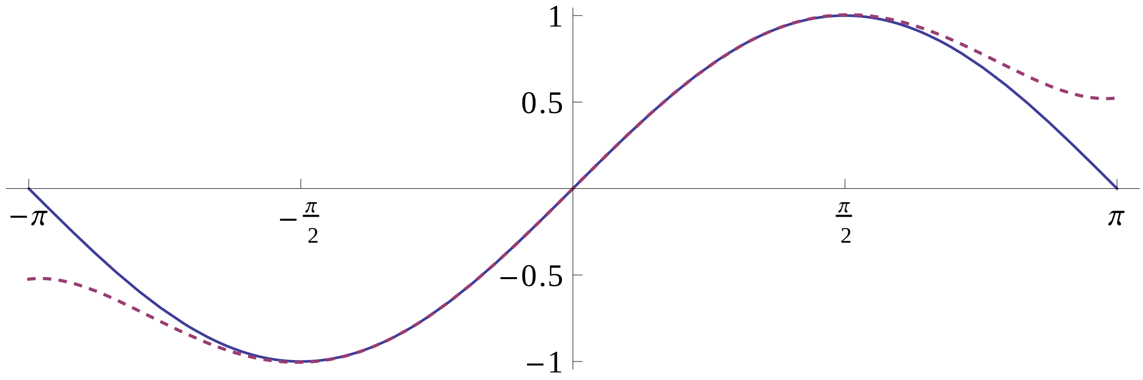

Figure 1: $y = \sin x$ and an approximation

equals zero at 0, 1 and 2 then $p(x)$ is a constant multiple of $x(x-1)(x-2)$. Now we know that $\sin x$ vanishes at 0, $\pm\pi$, $\pm 2\pi$ and so on so it should be the case that

$$\sin x = Ax(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)\ldots = Ax(x^2-\pi^2)(x^2-4\pi^2)\ldots$$

for some number A.

The problem with this is that each time we multiply by a further factor, we change the polynomial dramatically. For example when x is fairly small, the factor $x^2 - 9\pi^2$ is about $-9\pi^2 \approx -90$ so we make the polynomial much larger in size and change the sign. So there is no hope that this product will approach $\sin x$. Euler's solution was to write down a different product but with factors that vanish at the same places:

$$Ax\left(1 - x^2/\pi^2\right)\left(1 - x^2/(4\pi^2)\right)\left(1 - x^2/(9\pi^2)\right)\ldots$$

Now the successive factors get closer to 1 so it looks as though the product might approach something. In order to find out what should be the number A, imagine expanding the product. That means you pick one term from each bracket and multiply them together. If you pick the 1 from each bracket then the product is $Ax.1.1.1\ldots = Ax$. So in order to get the first term of expression in (1) we need to choose $A = 1$. So Euler guessed that

$$\sin x = x\left(1 - x^2/\pi^2\right)\left(1 - x^2/(4\pi^2)\right)\left(1 - x^2/(9\pi^2)\right)\ldots$$

which we know from equation (1) is the same as

$$x - x^3/6 + x^5/120 - \cdots = x\left(1 - x^2/\pi^2\right)\left(1 - x^2/(4\pi^2)\right)\left(1 - x^2/(9\pi^2)\right)\ldots$$

Now again imagine expanding the product but this time, choose an x^2 term from one of the brackets and 1 from each of the other brackets. You will get various different terms: $-x^3/\pi^2$, $-x^3/(4\pi^2)$, $-x^3/(9\pi^2)$ and so on, depending upon which bracket you choose the x^2 term from. Putting them all together, you find that the x^3 term on the right side is

$$-\frac{x^3}{\pi^2}\left(1+\frac{1}{4}+\frac{1}{9}+\cdots\right)$$

The x^3 term on the left side is $-x^3/6$ and, assuming that they must be the same, we get Euler's discovery

$$1+\frac{1}{4}+\frac{1}{9}+\cdots=\frac{\pi^2}{6}.$$

To make this argument precise would be beyond the scope of these notes but Euler's idea can be implemented in a different way that uses less advanced mathematics. Instead of pretending that $\sin x$ looks like a polynomial we shall choose a particular family of polynomials and apply Euler's argument to them. If these polynomials look like $\sin x$ then presumably whatever formula we derive will enable us to show that

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=\frac{\pi^2}{6}.$$

But, not only do we get to choose which polynomials to use (so we can choose some that work nicely), we don't need to *prove* that they look like $\sin x$. Just as long as they do, they should give us a useful formula.

Before embarking on this plan, let's collect together what Euler's argument really says. If we have a polynomial with zeroes at $0, \pm z_1, \pm z_2, \ldots, \pm z_m$ then we can write it in the form

$$Ax(1-x^2/z_1^2)(1-x^2/z_2^2)\ldots(1-x^2/z_m^2).$$

If we also know that this polynomial is

$$Ax-Bx^3+Cx^5-\cdots$$

then we can conclude that

$$\frac{1}{z_1^2}+\frac{1}{z_2^2}+\cdots+\frac{1}{z_m^2}=\frac{B}{A}.$$

3 The Chebyshev polynomials

Recall that

$$\cos 2\theta = 2\cos^2\theta - 1.$$

This says that the function $\cos 2\theta$ can be written as a polynomial function of $\cos\theta$. If we set $T_2(x) = 2x^2 - 1$ then

$$\cos 2\theta = T_2(\cos\theta).$$

In the same way we can use the addition formulae for cosine to write $\cos 3\theta$ in terms of $\cos\theta$.

$$\begin{aligned}
\cos 3\theta &= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta \\
&= \cos\theta(2\cos^2\theta - 1) - \sin\theta(2\sin\theta\cos\theta) \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta(\sin^2\theta) \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta) \\
&= 4\cos^3\theta - 3\cos\theta.
\end{aligned}$$

So if we define $T_3(x) = 4x^3 - 3x$ we get $T_3(\cos\theta) = \cos 3\theta$.

We can continue in this way to build up polynomials T_n which convert $\cos\theta$ into $\cos n\theta$. We missed out $T_1(x)$ which converts $\cos\theta$ to itself: so $T_1(x) = x$. These are called the Chebyshev polynomials. In order to simplfy the process of calculating them it is useful to have a general relationship. Note that

$$\begin{aligned}
\cos(n+1)\theta + \cos(n-1)\theta &= \cos n\theta\cos\theta - \sin n\theta\sin\theta + \cos n\theta\cos\theta + \sin n\theta\sin\theta \\
&= 2\cos n\theta\cos\theta.
\end{aligned}$$

Replacing $\cos\theta$ by x we get

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x).$$

This means we can write T_{n+1} in terms of T_n and T_{n-1}:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

For example $T_3(x) = 2xT_2(x) - T_1(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x$. Now we can calculate T_4, T_5 and so on:

$$T_4(x) = 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1$$

$$T_5(x) = 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x) = 16x^5 - 20x^3 + 5x.$$

Let's draw a graph of $T_3(x)$ for x between -1 and 1 (see Fig. 2). The graph shows that if $-1 \leq x \leq 1$ the image $T_3(x)$ also lies between -1 and 1. This is easy to

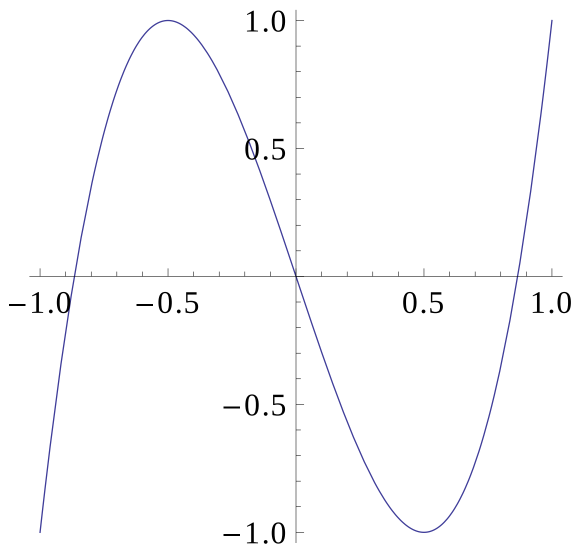

Figure 2: $y = T_3(x)$

understand from the formula $T_3(\cos\theta) = \cos 3\theta$. If x is between -1 and 1 then it is equal to $\cos\theta$ for some θ between 0 and π. So $T_3(x)$ is equal to $\cos 3\theta$ and this number is between -1 and 1. We can also see that T_3 has 3 zeroes in the interval. We can find where these are by solving the equation $4x^3 - 3x = 0$:

$$4x^3 - 3x = x(4x^2 - 3) = 0$$

if $x = 0$ or $x = \pm\sqrt{3}/2$. Notice that these numbers are $\cos\pi/2$, $\cos\pi/6$ and $\cos 5\pi/6$. To see why the zeroes have this form, let's choose $\cos\pi/6$ for example. If $\theta = \pi/6$ then $3\theta = \pi/2$ and $\cos 3\theta = \cos\pi/2 = 0$. This means that T_3 is equal to zero at $\cos\pi/6$:

$$T_3(\cos\pi/6) = \cos 3\pi/6 = \cos\pi/2 = 0.$$

Similarly $T_3(\cos\pi/2) = \cos 3\pi/2 = 0$ and $T_3(\cos 5\pi/6) = \cos 5\pi/2 = 0$.

Now let's look at T_5 (see Fig. 3). Notice that this graph looks rather like the graph of $\sin x$ but with a higher frequency. The polynomial vanishes at $\cos\pi/10$, $\cos 3\pi/10$, $\cos 5\pi/10$, $\cos 7\pi/10$ and $\cos 9\pi/10$ because, for example,

$$T_5(\cos\pi/10) = \cos\pi/2 = 0.$$

For our purposes it will be more covenient to express these numbers in terms of sine rather than cosine, using the fact that

$$\cos x = \sin(\pi/2 - x).$$

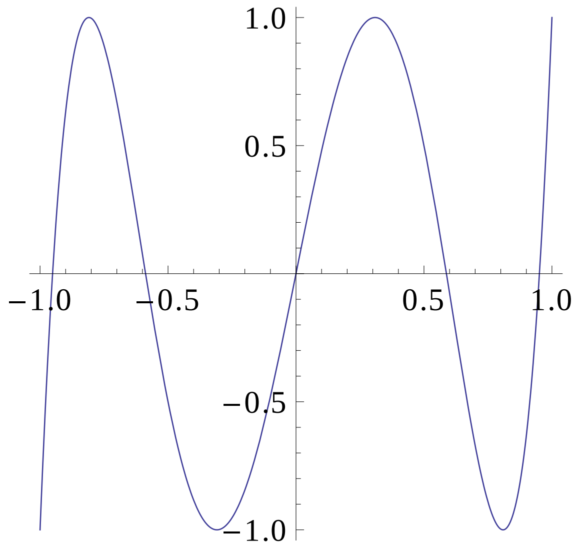

Figure 3: $y = T_5(x)$

Note that the middle one is $\cos 5\pi/10 = \cos \pi/2 = \sin 0$ which is zero. Similarly $\cos 3\pi/10 = \sin(\pi/2 - 3\pi/10) = \sin \pi/5$ and $\cos \pi/10 = \sin 2\pi/5$. The other zeroes $\cos 7\pi/10$ and $\cos 9\pi/10$ van be written as $-\sin \pi/5$ and $-\sin 2\pi/5$. If we copy Euler's argument for sine we can express T_5 as a product:

$$T_5(x) = Ax\left(1 - x^2/(\sin^2 \pi/5)\right)\left(1 - x^2/(\sin^2 2\pi/5)\right)$$

for some number A. So we now have an exact formula

$$16x^5 - 20x^3 + 5x = Ax\left(1 - x^2/(\sin^2 \pi/5)\right)\left(1 - x^2/(\sin^2 2\pi/5)\right)$$

and we can see by expanding the right side that $A = 5$. Now if we consider the x^3 term in the equation

$$16x^5 - 20x^3 + 5x = 5x\left(1 - x^2/(\sin^2 \pi/5)\right)\left(1 - x^2/(\sin^2 2\pi/5)\right)$$

we get

$$20 = 5\left(\frac{1}{\sin^2 \pi/5} + \frac{1}{\sin^2 2\pi/5}\right).$$

So we deduce that

$$\frac{1}{\sin^2 \pi/5} + \frac{1}{\sin^2 2\pi/5} = 4.$$

So, as we saw at the end of the last section, the sum of the reciprocals of the squares of the zeroes that we want is the ratio $20/5$ of the x^3 and x coefficients in the polynomial.

We can continue in this way to get simple expressions for

$$\frac{1}{\sin^2 \pi/7} + \frac{1}{\sin^2 2\pi/7} + \frac{1}{\sin^2 3\pi/7}$$
$$\frac{1}{\sin^2 \pi/9} + \frac{1}{\sin^2 2\pi/9} + \frac{1}{\sin^2 3\pi/9} + \frac{1}{\sin^2 4\pi/9}$$
$$\vdots$$
$$\frac{1}{\sin^2 \pi/n} + \frac{1}{\sin^2 2\pi/n} + \cdots + \frac{1}{\sin^2 m\pi/n}$$

and so on, for each odd number $n = 2m+1$. In order to find out what those expressions are we need to understand the x^3 and x terms of the Chebyshev polynomials.

The following is a table of the terms of the first few polynomials:

T_1		x								
T_2	-1		$2x^2$							
T_3		$-3x$		$4x^3$						
T_4	1		$-8x^2$		$8x^4$					
T_5		$5x$		$-20x^3$		$16x^5$				
T_6	-1		$18x^2$		$-48x^4$		$32x^6$			
T_7		$-7x$		$56x^3$		$-112x^5$		$64x^7$		
T_8	1		$-32x^2$		$160x^4$		$-256x^6$		$128x^8$	
T_9		$9x$		$-120x^3$		$432x^5$		$-576x^7$		$256x^9$

As you probably already spotted, the even numbered polynomials only involve even powers of x and the odd numbered ones only odd powers of x. In each column the non-zero coefficients alternate in sign as we move down the column. We are interested in the odd numbered ones and it is easy to guess from the table that the x coefficient of T_n is $\pm n$ if n is odd. In order to confirm this we can use the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

I leave this confirmation to the reader.

It is a bit more difficult to guess the formula for the x^3 terms but with a bit of trial and error you can come up with $n(n^2-1)/6$ if n is odd. This can again be confirmed using the recurrence relation. Once we have confirmed it we can calculate the ratio of the x^3 and x terms and we get $(n^2-1)/6$. At that point we have demonstrated the following formula: if $n = 2m+1$ is an odd number then

$$\sum_{k=1}^{m} \frac{1}{\sin^2(k\pi/n)} = \frac{n^2-1}{6}.$$

In some ways this formula may look even more remarkable than Euler's although I can't say I feel that way. What is pretty clear is that we can deduce the value of Euler's series quite quickly from the new formula. If we divide both sides by n^2 we get

$$\sum_{k=1}^{m} \frac{1}{(n\sin(k\pi/n))^2} = \frac{n^2-1}{6n^2}.$$

The right side approaches 1/6 as n gets larger so we need only show that the left side approaches

$$\sum_{1}^{\infty} \frac{1}{k^2\pi^2} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \cdots .$$

For each value of k the expression $n \sin k\pi/n$ approaches $k\pi$ as n gets larger. This by itself is not quite enough to solve the problem because there are possible pitfalls when dealing with infinite sums but in this case the pitfalls can be handled in a straightforward way.