SOME NOTES ON INEQUALITIES

NGOC MAI TRAN

Abstract. In each section of this note we investigate some useful inequalities, with emphasis on how we can apply them to solving problems and intuition on when to use what. Much of the materials are taken out of the excellent book 'Cauchy-Schwarz Masterclass' by Steele.

1. Cauchy-Schwarz inequality

This famous inequality comes in many forms (like all children of Cauchy are called Cauchy - it's impossible to tell on paper who's the 'real' one). So in this note we will name this guy

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n \leq \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}\sqrt{b_1^2 + b_2^2 + \ldots + b_n^2}$$

where $a_i, b_i \in \mathbb{R}$ the Cauchy's inequality for real numbers. Equality holds if and only if $a_i = \lambda b_i$ for all i, for some $\lambda > 0$.

Of course this guy carries many ID cards to prove himself: here's a proof (exercise 2. below). See Section 5 for hints to two other proofs.

1. Show that $ab + ac + bc \leq a^2 + b^2 + c^2$

2. Show that $\sum_{i=1}^n |a_ib_i| \leq \frac{1}{2}\sum_{i=1}^n a_i^2 + \frac{1}{2}\sum_{i=1}^n b_i^2$.
Now plug in a clever choice of a and b to obtain the Cauchy-Schwarz inequality.

In applying this inequality there are two major tricks: "1", and "splitting".

3. Show that for each real sequence $a_1, \ldots, a_n$, one has

$$a_1 + a_2 + \ldots + a_n \leq \sqrt{n}(a_1^2 + \ldots + a_n^2)^{1/2}$$

and

$$\sum_{k=1}^n a_k \leq (\sum_{k=1}^n |a_k|^{2/3})^{1/2}(\sum_{k=1}^n |a_k|^{4/3})^{1/2}.$$

4. Show that for all $x, y, z > 0$, one has

$$(\frac{x+y}{x+y+z})^{1/2} + (\frac{x+z}{x+y+z})^{1/2} + (\frac{y+z}{x+y+z})^{1/2} \leq 6^{1/2}$$

5. Show that for all $x, y, z > 0$, one has

$$x + y + z \leq 2\{\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\}.$$

Department of Statistics, UC Berkeley, CA 94703, USA
E-mail address: tran@stat.berkeley.edu.
Date: Nov 11, 2011.

6. Show that for $xyz \neq 0$,

$$\frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz} \leq \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}.$$

2. A bit of geometry: sum of squares and triangle inequality

You could think of Cauchy-Schwarz as a way to deal with *products* of sums of squares (SOS). What about *sums* of SOS? Note the geometric interpretation of $\sqrt{a_1^2 + \ldots + a_n^2}$ as the length of the vector $(a_1, \ldots, a_n)$. Here the low-tech *triangle inequality* can come to the rescue. Recall the triangle inquality: $|AC| \leq |AB| + |BC|$ for triangle ABC. In vector form, this states

$$\sqrt{(x_1+y_1)^2 + (x_2+y_2)^2} \leq \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2}$$

1. For $x, y, z \geq 0$, show that $(x+y+z)\sqrt{2} \leq \sqrt{x^2+y^2} + \sqrt{y^2+z^2} + \sqrt{x^2+z^2}$

2. For $x, y, z \geq 0$, show that

$$\sqrt{3} \leq \sqrt{x^2+y^2+z^2} + \sqrt{\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}}.$$

3. Prove the triangle inequality using Cauchy-Schwarz inequality.

3. Conversion between sums and products

As hinted in the proof of problem **1**, a close relative of Cauchy-Schwarz is the arithmetic-geometric mean *AM-GM inequality*:

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + \ldots + a_n}{n}$$

for all $a_1, a_2, \ldots a_n \geq 0$. Equality holds if and only if the a_i's are all equal.

Here is the reason why this inequality is called AM-GM

1a. Show that among all rectangles with a given perimeter, the square has the largest area.

1b. Show that among all boxes with a given surface area, the cube has the largest volume.

The AM-GM inequality generalizes itself.

2. Let $p_1, p_2, \ldots, p_n$ be nonnegative rational weights, $\sum_i p_i = 1$. Show that the 'general' AM-GM

$$a_1^{p_1} \ldots a_n^{p_n} \leq p_1 a_1 + \ldots + p_n a_n$$

follow from the plain vanilla AM-GM.

However, even if you love vanilla, don't underestimate the flavored AM-GM. It grants us the *splitting trick.* The one thing you should remember is that **AM-GM is very useful when we want to convert a product into a sum.**

3. Show that for $x, y \geq 0$, $x^{2011}y + y^{2011}x \leq x^{2012} + y^{2012}$

4. Show that for $x, y, z \geq 0$ $1 \leq xyz \Rightarrow 8 \leq (1+x)(1+y)(1+z)$.

5. Show that for $a, b, c > 0$,

$$a + b + c \leq \frac{a^3}{bc} + \frac{b^3}{ac} + \frac{c^3}{ab}.$$

When does equality hold?

6. For $a_k, b_k \geq 0$, $1 \leq k \leq n$, show that

$$(\prod_{k=1}^{n} a_k)^{1/n} + (\prod_{k=1}^{n} b_k)^{1/n} \leq (\prod_{k=1}^{n} (a_k + b_k))^{1/n}.$$

When does equality hold?

7. Show that for $x, y, z > 0$,

$$x^2(y^3 + z^3) + y^2(x^3 + z^3) + z^2(x^3 + y^3) \leq x(y^4 + z^4) + y(x^4 + z^4) + z(x^4 + y^4)$$

4. Convexity: the criminal behind the splitting trick

Many inequalities are derived by splitting 1 into a sum of nonnegative weights, such as the AM-GM inequality and its cousin seen above. This is more than a trick: it's a crime, and the criminal responsible for the inequality is often a 'convex' function.

Definition: A function $f : [a, b] \to \mathbb{R}$ is *convex* if for all $x, y \in [a, b]$ and $0 \leq p \leq 1$,

$$f(px + (1 - p)y) \leq pf(x) + (1 - p)f(y).$$

(What does this mean geometrically?) Now, convex functions satisfy *Jensen's inequality*:
If $f : [a, b] \to \mathbb{R}$ is a convex function, and p_j are nonnegative numbers summing to 1, then for all $x_j \in [a, b]$, $j = 1, \ldots, n$:

$$f(\sum_{j=1}^{n} p_j x_j) \leq \sum_{j=1}^{n} p_j f(x_j).$$

Equality holds when $x_1 = x_2 = \ldots = x_n$.
Here are some examples of important convex functions:

- The function $x \mapsto e^x$
- The function $x \mapsto x^n$ for any $n > 1$
- The function $x \mapsto 1/sin(x)$

Functions where the inequality goes 'the other way', that is, $f(px + (1 - p)y) \geq pf(x) + (1 - p)f(y)$ are called *concave* functions. Check that *if f is concave, then $1/f$ is convex.* This can be helpful in many inequalities with fractions.

1. Show that if $x > 1$, then

$$\frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1} > \frac{3}{x}$$

2. Show that if $x, y, z > 0$, $x + y + z = 1$, then

$$64 \leq (1 + \frac{1}{x})(1 + \frac{1}{y})(1 + \frac{1}{z})$$

3. Prove the AM-GM inequality using Jensen's inequality.

4. Let a, b, c be the sides of a triangle, A be its area. Show that

$$a^2 + b^2 + c^2 \geq \frac{4}{\sqrt{3}} A$$

When does equality hold?

5. Show that for $a_1, a_2, a_3, a_4 > 0$, one has

$$2 \leq \frac{a_1}{a_2 + a_3} + \frac{a_2}{a_3 + a_4} + \frac{a_3}{a_4 + a_1} + \frac{a_4}{a_1 + a_2}$$

5. Appendix: Proofs of inequalities introduced

Fill out the details of some main proofs below.

Proof of Cauchy-Schwarz inequality for real numbers.

(1) *Induction.* The case $n = 2$: square both sides and rearrange. For the induction step: split sum to two pieces, then apply the inequality.
(2) *Geometry.* (If you know about vector space and inner products): rewrite as an inequality involving inner products. Now use $\langle u - v, u - v \rangle \geq 0$, and plug in $u = a/\|a\|, v = b/\|b\|$. (How is this related to the proof hinted in problem **1**?

Proof of the AM-GM inequality

(1) *Induction.* Prove by induction on $n = 2^k$. For $2^k < n < 2^{k+1}$, padd the original sequence with $A = \frac{a_1 + \ldots a_n}{n}$ and reduce to the case $n = 2^{k+1}$.
(2) *Polya's dream.* Use the fact that the function $x \mapsto e^x$ is convex. According to the book by Steele, Polya discovered this proof in a dream and reported it as the best mathematics he had ever dreamt.

Proof of Jensen's inequality: induction.