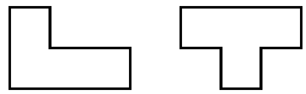
Berkeley Math Circle
Monthly Contest 8 – Solutions

1. Determine whether it is possible to tile a standard 8×8 chessboard with 15 L-tiles and 1 T-tile of the shapes below:



(Each tile covers four squares of the chessboard. The tiles can be flipped and rotated at will.)

Solution. Let the squares of the chessboard be colored black and white in the usual way. Note that the L-tile always covers two squares of each color, while the T-tile covers either 1 black and 3 white or 3 black and 1 white squares. Since the chessboard has equal numbers of black and white squares, it is impossible to cover it with these tiles.

2. A point P lies inside a regular hexagon $ABCDEF$. The distances from P to the sides AB, BC, CD, DE, EF, and FA are respectively 1, 2, 5, 7, 6, and x. Find x.

Solution. Opposite sides of a regular hexagon are parallel. The sum of the distances from P to AB and to DE is simply the distance between AB and DE, which must therefore equal $1 + 7 = 8$. Because of the symmetry of the hexagon, the distance between any pair of opposite sides must be 8. Applying this to sides CD and FA, we see that the distance from P to CD is 5 so the distance from P to FA must be $8 - 5 = 3$.

3. Let k be a positive integer. Prove that there is a positive integer N with the following properties:

   (a) N has k digits, none of which is 0.

   (b) No matter how the digits of N are rearranged, the resulting number is not divisible by 13.

Solution. Let N_1 be the number consisting of k ones. If N_1 is not divisible by 13, it is the number N we seek, since no matter how its digits are rearranged, it is still the same number.

If N_1 is divisible by 13, consider $N_1 + 1$, that is to say the number consisting of $k - 1$ ones and one 2. When the digits of this number are rearranged, the result has the form

$$11 \cdots 121 \cdots 11 = 11 \cdots 111 \cdots 11 + 10 \cdots 00 = N_1 + 10^k,$$

where $k \geq 0$ is an integer. This is not divisible by 13 since N_1 is divisible by 13 and 10^k is not.

4. Suppose that b and c are real numbers such that the equation

$$x^2 + bx + c = 0$$

has two different solutions x_1, x_2. Suppose that

   (a) The (positive) difference between x_1 and x_2 is 1;

   (b) The (positive) difference between b and c is also 1.

Find all possible values of b and c.

Solution. Given the two solutions x_1 and x_2, we know that the equation factors:

$$x^2 + bx + c = (x - x_1)(x - x_2) = 0.$$

From this we derive that

$$b = -x_1 - x_2 \text{and} c = x_1 x_2.$$

In condition (a), we switch the labels on x_1 and x_2, if necessary, to assume that $x_2 = x_1 + 1$. However, for condition (b) we still have the two cases $c - b = 1$, $c - b = -1$.

We compute

$$\begin{aligned} c - b &= x_1 x_2 + x_1 + x_2 \\ &= x_1(x_1 + 1) + x_1 + (x_1 + 1) \\ &= x_1^2 + 3x_1 + 1. \end{aligned}$$

Setting $c - b = 1$ gives $x_1^2 + 3x_1 = 0$, which has two solutions:

(a) $x_1 = 0$, giving $x_2 = 1$, $b = -1$, $c = 0$;

(b) $x_1 = -3$, giving $x_2 = -2$, $b = 5$, $c = 6$.

Likewise, setting $c - b = -1$ gives $x_1^2 + 3x_1 + 2 = 0$, which also has two solutions:

(a) $x_1 = -1$, giving $x_2 = 0$, $b = 1$, $c = 0$;

(b) $x_1 = -2$, giving $x_2 = -1$, $b = 3$, $c = 2$.

5. Prove that any prime which is the difference of two cubes is also the sum of a square and three times a square.

Remark. By *cube* and *square* are meant, respectively, the cube and square of a natural number.

Solution. Suppose that

$$p = a^3 - b^3$$

where p is prime and a and b are natural numbers. The right side can be factored:

$$p = (a - b)(a^2 + ab + b^2)$$

Since p is prime, one of the factors must be ± 1. We note that $a^2 + ab + b^2 \geq 1 + 1 + 1 = 3$, so $a - b = 1$ and

$$p = a^2 + ab + b^2 = (b + 1)^2 + b(b + 1) + b^2 = 3b^2 + 3b + 1.$$

Trying to express $3b^2 + 3b + 1$ directly as $f(b)^2 + 3g(b)^2$, where $f(b)$ and $g(b)$ are integer polynomials, leads to difficulties (the coefficient of b can never be made odd). However, we note that b is either even or odd. If $b = 2x$, then

$$p = 12x^2 + 6x + 1 = (3x + 1)^2 + 3x^2,$$

while if $b = 2x - 1$, then

$$p = 12x^2 - 6x + 1 = (3x - 1)^2 + 3x^2.$$

Thus $p = c^2 + 3d^2$ where c and d are integers.

6. Given five nonnegative real numbers with sum 1, prove that it is possible to arrange them at the vertices of a regular pentagon such that no two numbers connected by a side of the pentagon have product exceeding $1/9$.

Solution. Label the numbers a, b, c, d, e in increasing order. Place them around the pentagon in the order e, a, d, c, b. Then it is clear that the products of the numbers on the sides follow the inequalities

$$ad \leq ae \leq be \quad \text{and} \quad bc \leq cd.$$

Thus it suffices to prove that $be \leq 1/9$ and $cd \leq 1/9$. Using the AM-GM inequality,

$$1 = a + b + c + d + e \geq 0 + 0 + c + d + d = c + 2d \geq \frac{3c}{2} + \frac{3d}{2} \geq 2\sqrt{\frac{3c}{2} \cdot \frac{3c}{2}} = 3\sqrt{cd},$$

so $cd \leq 1/9$. Also,

$$1 = a + b + c + d + e \geq 0 + b + b + b + e = 3b + e \geq 2\sqrt{3b \cdot e}$$

so $be \leq 1/12 < 1/9$.

7. In triangle ABC, $\angle A = 60°$. Let E and F be points on the extensions of AB and AC such that $BE = CF = BC$. The circumcircle of ACE intersects EF in K (different from E). Prove that K lies on the bisector of $\angle BAC$.

Solution. Let the bisector of $\angle BAC$ intersect the circumcircle of $\triangle ACE$ at K'. The arcs, and hence the chords, $K'C$ and $K'E$ are equal; since $CB = BE$ is given, we have $\triangle K'BC \cong \triangle K'BE$ and so K' is on the bisector of $\angle CBE$. This shows that K' is the excenter of $\triangle ABC$ opposite A. By symmetry, we could have defined K' as the intersection of the bisector of $\angle BAC$ and the circumcircle of $\triangle ABF$, and it would have been the same excenter.

To prove that $K' = K$, it remains to show that F, K', and E are collinear. Since $\angle CK'E = \angle BK'F = 120$ (by cyclic quads $ACK'E$ and $ABK'F$, it suffices to show that $\angle BK'C = 60$. But this follows from the properties of the excenter:

$$\begin{aligned}\angle BK'C &= 180 - \angle CBK' - \angle K'CB \\ &= 180 - \frac{1}{2}\angle CBE - \frac{1}{2}\angle FCB \\ &= 180 - \frac{360 - \angle ABC - \angle BCA}{2} \\ &= 180 - \frac{180 + \angle CAB}{2} \\ &= 180 - \frac{180 + 60}{2} = 60.\end{aligned}$$