Berkeley Math Circle
Monthly Contest 7
Due April 3, 2012

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 5
by Bart Simpson
in grade 7, BMC Intermediate
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

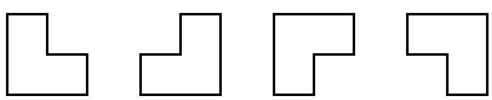
Enjoy solving these problems and good luck!

Problems

1. Prove that
$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$
for all positive integers n.

2. Do there exist four consecutive positive integers whose product is a perfect square?

3. Fix a positive integer n. Two players, Phil and Ellie, play the following game. First, Phil fills the squares of an $n \times n$ chessboard with nonnegative integers less than n. Then, Ellie chooses three squares making an L, as in any of the following pictures:



Ellie adds 1 to each of the three squares making the L, except that if the number n appears in a square, it is immediately replaced by 0. Ellie wins if, after modifying finitely many L's in this way, she can change all the numbers on the board into 0's; otherwise Phil wins.

Which player has a winning strategy if

(a) $n = 12$?

(b) $n = 2012$?

4. For a positive integer n, let $f(n)$ be the number of divisors of n which are perfect squares, and let $g(n)$ be the number of divisors of n which are perfect cubes. Determine whether there exists an integer n such that
$$\frac{f(n)}{g(n)} = 2012.$$

5. Let ABC be a triangle with incenter I. The circumcircle of $\triangle AIB$ meets the lines CA and CB again at P (different from A) and Q (different from B) respectively. Prove that A, B, P, and Q are (in some order) the vertices of a trapezoid.

6. Determine all positive integers n such that there exist n distinct three-element subsets $A_1, A_2, \ldots, A_n$ of the set $\{1, 2, \ldots, n\}$ such that $|A_i \cap A_j| \neq 1$ for all i and j, $1 \leq i < j \leq n$.

7. Consider the function

$$f(x) = \frac{(x-2)(x+1)(2x-1)}{x(x-1)}.$$

Suppose that u and v are real numbers such that

$$f(u) = f(v).$$

Suppose that u is rational. Prove that v is rational.