Berkeley Math Circle
Monthly Contest 5
Due March 6, 2012

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 5
by Bart Simpson
in grade 7, BMC Intermediate
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

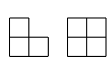
1. Is
$$\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\cdots+\frac{2011}{2012}$$
an integer? Prove your answer.

2. At a market, a buyer and a seller each have four exotic coins. You are allowed to label each of the eight coins with any positive integer value in cents. The labeling is called n-*efficient* if for any integer k, $1 \leq k \leq n$, it is possible for the buyer and the seller to give each other some of their coins in such a way that, as a net result, the buyer has paid the seller k cents. Find the greatest positive integer n such that an n-efficient labeling of the coins exists.

3. Let $ABCD$ be a square in the coordinate plane such that A is on the x-axis and C is on the y-axis. Prove that one of the vertices B and D lies on the line $y = x$.

4. Let a, b, c, x be real numbers such that
$$ax^2 - bx - c = bx^2 - cx - a = cx^2 - ax - b.$$
Prove that $a = b = c$.

5. A 9×7 rectangle is tiled using only the two types of tiles below (the L-tromino and the 2×2 square):



They may be used in any orientation. Let s be the number of 2×2 squares in such a tiling; find all possible values of s.

6. On a quiz, every question is solved by exactly four students, every pair of questions is solved by exactly one student, and none of the students solved all of the questions. Find the maximum possible number of questions on the quiz.

7. In acute triangle ABC, the exterior angle bisector of $\angle BAC$ meets ray BC at D. Let M be the midpoint of side BC. Points E and F lie on the line AD such that $ME \perp AD$ and $MF \perp BC$. Prove that
$$BC^2 = 4AE \cdot DF.$$