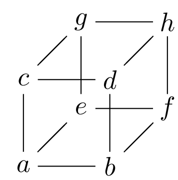
Berkeley Math Circle
Monthly Contest 5 – Solutions

1. Each vertex of a cube is labeled with an integer. Prove that there exist four coplanar vertices the sum of whose numbers is even.
Solution. Label the vertices of the cube as shown. The quadruples of vertices

$$\begin{array}{ccccccc} a & \text{and} & b & \text{and} & c & \text{and} & d \\ a & \text{and} & b & \text{and} & e & \text{and} & f \\ c & \text{and} & d & \text{and} & e & \text{and} & f \end{array}$$

are all coplanar. Assume that all three quadruples have an odd sum; then the sum of all twelve terms is odd. But these twelve terms add to



$$\begin{aligned}&(a+b+c+d)+(a+b+e+f)+(c+d+e+f)\\&=2(a+b+c+d+e+f),\end{aligned}$$

an even number. Contradiction!

2. A 2012×2012 table is to be filled with integers in such a way that each of the 4026 rows, columns, and main diagonals has a different sum. What is the smallest number of distinct values that must be used in the table?

Solution. Answer: 3.

If at most two numbers are used, say x and y, the sum of every row and column is completely determined by the number of y's it has, which ranges from 0 to 2012. Thus there are only 2013 possible sums, not enough for the 4026 rows, columns, and diagonals.

On the other hand, if n is a large integer (say, 10000), then it is possible to fill the table using 0, 1, and n. Here is a 6×6 example that generalizes readily:

$$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & n \\ 1 & 1 & 1 & 0 & n & n \\ 1 & 1 & 0 & n & n & n \\ 1 & 0 & 0 & 0 & n & n \\ 0 & 0 & 0 & 0 & 0 & n \end{array}$$

3. Let x, y, and z be real numbers.

   (a) Prove that
   $$x + y > |x - y|$$
   if and only if x and y are both positive.
   Solution. Note that $|x-y|$ is the greater of $x-y$ and $-(x-y)$. So the above assertion is equivalent to
   $$\begin{array}{ccc} x+y > x-y & \text{and} & x+y > -x+y \\ 2y > 0 & \text{and} & 2x > 0 \\ y > 0 & \text{and} & x > 0. \end{array}$$

   (b) Find an inequality involving the variables x, y, and z, using only the operations of addition, subtraction, multiplication and absolute value, that is true if and only if x, y, and z are all positive.
   Solution 1. The quantity
   $$x + |x|$$
   is $2x$, a positive number, if $x > 0$ and is 0 otherwise. Therefore the inequality
   $$(x + |x|)(y + |y|)(z + |z|) > 0$$
   holds if and only if x, y, and z are all positive.

Solution 2. By part (a), the condition that x, y, and z are all positive is equivalent to

$$x + y - |x - y| > 0 \quad \text{and} \quad z > 0.$$

By part (a) again, this is equivalent to

$$x + y - |x - y| + z > \big| x + y - |x - y| - z \big|.$$

This inequality involves no multiplication.

4. The tangents at A and B to the circumcircle of an acute triangle ABC intersect at T. Point D lies on line BC such that $DA = DC$. Prove that $TD \parallel AC$.

Solution. Using the properties of tangent lines and chords, we have

$$\angle TAB = \angle ACB = \angle ABT = \alpha.$$

Moreover, $\triangle DAC$ is isosceles so $\angle DAC = \alpha$ as well. Now triangles TAB and DAC are similar, so

$$\frac{AT}{AB} = \frac{AD}{AC} \quad \text{and} \quad \angle TAD = \angle BAC,$$

implying that triangles TAD and BAC are similar. Therefore

$$\angle ADT = \angle ACB = \alpha = \angle DAC,$$

so $TD \parallel AC$.

5. A grasshopper begins at the vertex $(1, 0)$ of an infinite lattice with origin $O(0, 0)$. It can jump from a point A to any lattice point B such that $\triangle OAB$ has area $1/2$.

   (a) Find all points (x, y) of the lattice which the grasshopper can reach.

   (b) Prove that if the grasshopper can reach a point (x, y), then it can reach it in at most $|y| + 2$ jumps, starting from $(1, 0)$.

Solution. The answer to part (a) is all pairs (x, y) of integers that are relatively prime.

If a point X has coordinates with a common factor, say (ku, kv), then there exists a lattice point $M(u, v)$ on the segment OX. Then for any lattice point A, the area of $\triangle OAX$ is not $1/2$ because it is k times the area of $\triangle OAM$. Therefore the grasshopper can never reach such a point X.

We will now prove that if x and y are relatively prime and nonnegative, then the grasshopper can reach (x, y) in at most y steps. As base cases, we note that can reach $(1, 0)$ in zero jumps and any point of the form $(x, 1)$, $x \in \mathbb{Z}$, in one jump. We continue by (strong) induction on y. We distinguish two cases for a given pair (x, y):

(1) If $y > x$, then by the induction hypothesis, (y, x) is attainable in at most x steps. We jump first from $(1, 0)$ to $(0, 1)$ and then apply the moves from $(1, 0)$ to (y, x) in reflected fashion. We get a solution using at most $x + 1 \le y$ steps.

(2) If $y < x$ (note that y cannot equal x if $y \ge 2$) then we consider the transformation of the plane sending a general point (u, v) of the plane to $(u - v, v)$. This sends $(1, 0)$ to itself, lattice points to lattice points, and lines to lines, and in fact it preserves the areas of triangles (this can be seen using either Pick's formula or the determinant formula for the area of a coordinate triangle). So (x, y) is attainable in a given number of steps if and only if $(x - y, y)$ is. We continue subtracting y from the x-coordinate until it is less than y (of course it will still be relatively prime to y), and then we apply the previous case.

The points (x, y) where $y < 0$ and $x \ge 0$ can be dealt with symmetrically. Finally, if x is negative, we jump from $(1, 0)$ to $(0, 1)$ to $(-1, 0)$, making two moves, and apply at most $|y|$ moves to get from $(-1, 0)$ to (x, y).

6. Determine whether there exist two distinct finite subsets A and B of the reals such that for every polynomial f of degree 2012 with real coefficients,

$$\sum_{x \in A} f(x) = \sum_{x \in B} f(x).$$

Solution. We will prove by induction on $n \geq 0$ that there exist distinct sets A_n and B_n satisfying the desired equality

$$\sum_{x \in A_n} f(x) = \sum_{x \in B_n} f(x) \tag{1}$$

for all polynomials f of degree *at most* n.

The base case is $n = 0$; we take $A = \{0\}$ and $B = \{1\}$. Now we suppose A_{n-1} and B_{n-1}, $n \geq 1$, to be constructed and define

$$A_n = A_{n-1} \cup (B_{n-1} + t)$$
$$B_n = B_{n-1} \cup (A_{n-1} + t)$$

where the parameter t of the translations is a real number large enough that the unions are actually disjoint unions. Now the condition (1) becomes

$$\sum_{x \in A_{n-1}} f(x) + \sum_{x \in B_{n-1}} f(x+t) = \sum_{x \in B_{n-1}} f(x) + \sum_{x \in A_{n-1}} f(x+t)$$
$$\sum_{x \in A_{n-1}} [f(x) - f(x+t)] = \sum_{x \in B_{n-1}} [f(x) - f(x+t)].$$

If $f(x)$ has degree at most n, then the polynomial $g(x) = f(x) - f(x+t)$ has degree at most $n-1$ since the leading terms of $f(x)$ and $f(x+t)$ are identical. Thus this last condition holds by the induction hypothesis.

7. Find all primes p such that there exist integers a, b, c, and k satisfying the equations

$$a^2 + b^2 + c^2 = p$$
$$a^4 + b^4 + c^4 = kp.$$

Solution. The answers are 2 and 3. It is clear that for them we can use the values $a = b = k = 1$ and $c = 0$ or 1 respectively.

Assume $p \geq 5$. Subtracting twice the second equation from the square of the first, we find that p divides

$$2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4,$$

an expression which factors as

$$(a+b+c)(a+b-c)(a-b+c)(-a+b+c).$$

One of the four factors must be divisible by p. By flipping the signs on a, b, and c, we can assume it is the first one. Using the inequality $x^2 - x \geq 0$ (for integers x, equality holding at $x = 0$ and $x = 1$), we have

$$p \leq a+b+c \leq a^2+b^2+c^2 = p.$$

This equality condition can only hold if a, b, and c are each equal to their squares, implying that they are at most 1 and $p \leq 1+1+1 = 3$.