Berkeley Math Circle
Monthly Contest 4
Due January 10, 2012

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 4
by Bart Simpson
in grade 7, BMC Intermediate
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. On a given street, there are n houses numbered from 1 to n. Let a_i $(1 \leq i \leq n)$ be the number of people living in the house numbered i, and let b_i $(i \geq 1)$ be the number of houses on the street in which at least i people live. Prove that
$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + b_3 + \cdots .$$
2. Let ABC be an isosceles right triangle with $\angle C = 90°$. Let X be a point on the line segment AC, and let Y and Z be, respectively, the feet of the perpendiculars from A and C to the line BX. Prove that
$$BZ = YZ + AY.$$
3. Determine, with proof, whether the following statement is true or false: Out of any six natural numbers, one can find either three which are pairwise relatively prime or three whose greatest common divisor is greater than 1.
4. A $2 \times n$ grid has a light bulb in each square. Each bulb has a switch that flips the state of its corresponding bulb as well as all (horizontally or vertically) adjacent bulbs. Determine whether it is always possible, regardless of the initial state of the bulbs, to turn all the bulbs off if
   (a) $n = 2011$
   (b) $n = 2012$.
5. Determine all pairs (n, k) of integers such that $0 < k < n$ and
$$\binom{n}{k-1} + \binom{n}{k+1} = 2\binom{n}{k}.$$
6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(xy + z) = f(x)f(y) + f(z)$$
for all real numbers x, y, and z.
7. Let ABC be a triangle. The incircle, centered at I, touches side BC at D. Let M be the midpoint of the altitude from A to BC. Lines MI and BC meet at E. Prove that $BD = CE$.