Berkeley Math Circle
Monthly Contest 3
Due December 6, 2011

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 3
by Bart Simpson
in grade 7, BMC Intermediate
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Let a and b be integers such that
$$|a+b| > |1+ab|.$$
Prove that $ab = 0$.

2. Let p be a prime number. Find all possible values of the remainder when $p^2 - 1$ is divided by 12.

   Remark. To receive the full 7 points, your solution must provide:

   (a) A list of the possible remainders;

   (b) Demonstrations that the numbers on your list can be remainders when $p^2 - 1$ is divided by 12;

   (c) A proof that numbers *not* on your list cannot be remainders.

3. We are given a 13×13 chessboard. Determine whether it is possible to place nonoverlapping 1×4 rectangular tiles on the board so as to cover every square but the central one.

4. Let $ABCD$ be a parallelogram. Suppose that the circumcenter of $\triangle ABC$ lies on diagonal BD. Prove that $ABCD$ is either a rectangle or a rhombus (or both).

5. Let $\ominus$ be an operation on the set of real numbers such that

$$(x \ominus y) + (y \ominus z) + (z \ominus x) = 0$$

for all real x, y, and z. Prove that there is a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$x \ominus y = f(x) - f(y)$$

for all real x and y.

6. Let N be a positive integer such that N is divisible by 81 and the number formed by reversing the digits of N is also divisible by 81. Prove that the sum of the digits of N is divisible by 81.

7. Let k be a positive integer, and let $(a_1, a_2, \ldots, a_{2k})$ and $(b_1, b_2, \ldots, b_{2k})$ be two sequences of real numbers such that $1/2 \leq a_1 \leq \cdots \leq a_{2k}$ and $1/2 \leq b_1 \leq \cdots \leq b_{2k}$. Let M and m be the maximum and minimum respectively of

$$(a_1 + c_1)(a_2 + c_2) \cdots (a_{2k} + c_{2k})$$

as $(c_1, \ldots, c_{2k})$ ranges through all possible permutations of $(b_1, \ldots, b_k)$. Prove that

$$M - m \geq k(a_k - a_{k+1})(b_k - b_{k+1}).$$