Berkeley Math Circle
Monthly Contest 2
Due November 1, 2011

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 2
by Bart Simpson
in grade 7, BMC Intermediate
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Find the number of multiples of 3 which have six digits, none of which is greater than 5.

   Remark. To receive the full 7 points, your solution must be proved; it must include not just the numerical answer and the calculations by which you found it, but also the reasons why those calculations give the answer to the problem.

2. On an infinite chessboard, two squares are said to *touch* if they share at least one vertex and they are not the same square. Suppose that the squares are colored black and white such that

   - there is at least one square of each color;
   - each black square touches exactly m black squares;
   - each white square touches exactly n white squares

   where m and n are integers. Must m and n be equal?

   Remark. If you think the answer is *yes*, you must prove it rigorously. If you think the answer is *no*, you should give values of m and n, as well as a tiling of the whole chessboard, to demonstrate that m and n can be unequal.

3. Is there an integer x such that

$$2010 + 2009x + 2008x^2 + 2007x^3 + \cdots + 2x^{2008} + x^{2009} = 0?$$

4. Let $ABCD$ be a convex quadrilateral such that $\angle ABD = \angle ACD$. Prove that $ABCD$ can be inscribed in a circle.

   Remark. Be especially rigorous in this problem. If you talk about the intersection of two lines, first prove that they are not parallel; if you mention the two intersections of a line and a circle, consider that they might be tangent; and so on.

5. Let $n > 3$ be a positive integer. Define an integer k to be *snug* if $1 \leq k < n$ and

$$\gcd(k, n) = \gcd(k+1, n).$$

Prove that the product of all snug integers is congruent to 1 modulo n.

Remark. If there are no snug integers, their product is vacuously declared to equal 1.

6. Let $ABCD$ be a convex quadrilateral. Suppose that the area of $ABCD$ is equal to

$$\frac{AB + CD}{2} \cdot \frac{AD + BC}{2}.$$

Prove that $ABCD$ is a rectangle.

7. Let N be a positive integer. Define a sequence a_n, $n \geq 0$, by

$$a_0 = 0, \qquad a_1 = 1, \qquad a_{n+1} + a_{n-1} = a_n \left(2 - \frac{1}{N}\right) \quad (n \geq 1).$$

Prove that $a_n < \sqrt{N+1}$ for all $n \geq 0$.