Berkeley Math Circle
Monthly Contest 1
Due October 4, 2011

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 1
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Let x be an odd positive integer other than 1. Prove that one can find positive integers y and z such that

$$x^2 + y^2 = z^2.$$

Remark. To receive the full 7 points, your solution must be rigorously proved; for instance, it must include:

- How y and z are found, given x;
- An explanation of why the y and z that you have found are positive integers;
- An explanation of why $x^2 + y^2 = z^2$.

2. A finite number of points are drawn in the plane. Prove that one can select two of them, A and B, such that:

(a) A and B are not the same point.

(b) No drawn point, other than A itself, is closer to A than B is.

(c) No drawn point, other than B itself, is closer to B than A is.

Rigorously prove your solution.

3. The number 2011 is written on a blackboard. It is permitted to transform the numbers on it by two types of moves:

   (a) Given a number n, we can erase n and write two nonnegative integers a and b such that $a + b = n$.

   (b) Given two numbers a and b, we can erase them and write their difference $a - b$, assuming this is positive or 0.

   Is it possible that after a sequence of such moves, there is only one number on the blackboard and it is 0? (The numbers must be treated as the numbers that they are, so it is not permissible, say, to write the numbers 1 and 2 and then reinterpret them as the number 12.)

   Remark. If you think that the answer is *yes*, clearly describe each move that must be performed in order to get from 2011 to 0. If you think that the answer is *no*, you must rigorously prove that it is impossible.

4. Let X, Y, and Z be points on one side of a segment AB such that

$$\triangle XAB \sim \triangle BYA \sim \triangle ABZ.$$

   Prove that $\triangle XYZ$ is similar to all these triangles.

5. For positive integers n, define

$$a_n = 3^n + 6^n - 2^n.$$

   Find, with proof, all primes that do not divide any of the numbers $a_1, a_2, a_3, \ldots$.

6. Let R be the region consisting of all points inside or on the boundary of a given circle of radius 1. Find, with proof, all positive real numbers d such that it is possible to color each point of R red, green or blue such that any two points of the same color are separated by a distance less than d.

7. Let $n > 1$ be an integer. Three complex numbers have the property that their sum is 0 and the sum of their nth powers is also 0. Prove that two of the three numbers have the same absolute value.

   Remark. The absolute value of a complex number is its distance from the origin, $|a + bi| = \sqrt{a^2 + b^2}$.