The Pigeonhole Principle

Berkeley Math Circle - Beginner's

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Many of these problems are from *Mathematical Circles (Russian Experience)* and from *A Decade of the Berkeley Math Circle -Volume 1*

A. (BAMO 2010 Problem 4) Place eight rooks on a standard 8×8 chessboard so that no two are in the same row or column. With the standard rules of chess, this means that no two rooks are attacking each other. Now paint 27 of the remaining squares (not currently occupied by rooks) red. Prove that no matter how the rooks are arranged and which set of 27 squares are painted, it is always possible to move some or all of the rooks so that:

- All the rooks are still on unpainted squares.
- The rooks are still not attacking each other (no two are in the same row or same column).
- At least one formerly empty square now has a rook on it; that is, the rooks are not on the same 8 squares as before.

B. (BAMO 2005 Problem 3) Let n be an integer greater than 12. Points $P_1, P_2, \ldots, P_n, Q$ in the plane are distinct. Prove that for some i , at least $n/6 - 1$ of the distances

$$P_1P_i, P_2P_i, \ldots, P_{i-1}P_i, P_{i+1}P_i, ..., P_nP_i$$

are less than P_iQ.

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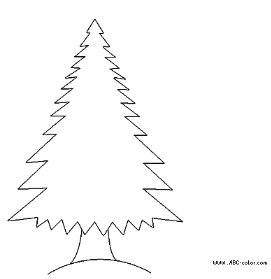
1. I own 7 pairs of socks and each pair is a different color. If all 14 socks are loose in the dryer, how many will I have to pull out to guarantee that I get at least two of the same color?

2. Over a million Christmas trees were sold in California this winter. No tree has more than 800,000 needles on it. Show that two of the Christmas trees had the same number of needles on them at midnight of the night before Christmas.



3. A bag contains 10 black marbles and and 10 white marbles. What is the smallest number of marbles that you must pull out to guarantee that you get at least two marbles of the same color?

4. Given 12 integers, show that two of them can be chosen whose difference is divisible by 11.

5. The population of the Bay Area is about 7.4 million people. Show that at least two people in California have the exact same number of hairs on their heads. The average number of hairs on a human head is about 100,000. Assume that no person has more than a million hairs on their heads.

6. Sixteen boxes of chocolate are for sale at the store. The chocolates are of three different kinds (dark, milk chocolate, and white chocolate), and all chocolates in a box are of the same kind. You want to buy 6 boxes of chocolates to give to your 6 cousins, but you want to give them all the same kind of chocolate so there won't be any squabbling. Is this necessarily possible?

Pigeonhole Principle:

a) If you put $n+1$ or more pigeons into n pigeon holes, at least one pigeon hole must contain more than one pigeon.

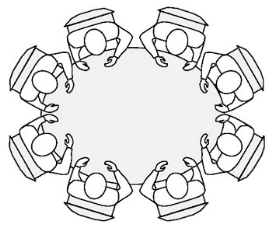
b) If you put $kn+1$ or more pigeons into n pigeon holes, at least one pigeon hole must contain more than k pigeons.

7. Given eight different positive integers, none greater than 15, show that at least three pairs of them have the same positive difference. (The pairs may overlap – that is, two pairs or all three pairs may contain the same integer.)

8. Show that in any group of five people, there are two who have an identical number of friends within the group. (Friendship is mutual – if A is B's friend, then B is A's friend.)

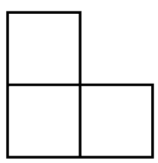
9. Several soccer teams enter a tournament in which each team plays every other team exactly once. Show that, at any moment during the tournament, there will be two teams which have played, up to that moment, the same number of games.

10. Eight chairs are set around a circular table. On the table are name placards for eight guests. After the guests are seated, it is discovered that none of them are in front of their own names. Show that the table can be rotated so that at least two guests are simultaneously correctly seated.



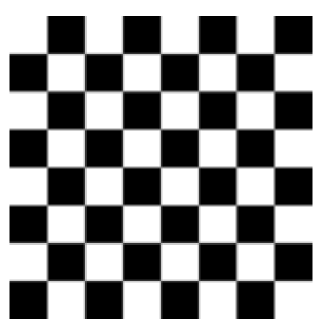
11. Fifty-one points are scattered within a square of side length one meter. Show that at least 3 of the points can be covered with a square of side length 20 cm.

12. What is the largest number of squares on an 8×8 checkerboard which can be colored green, so that in any arrangement of three squares (a ”tromino” as drawn below), at least one square is not colored green? (The tromino may be appear as in the figure or it may be rotated through some multiple of 90 degrees.)



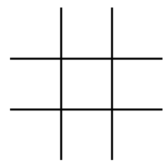
13. What is the smallest number of squares which can be colored green, so that in any tromino at least one square is colored green?

14. What is the largest number of kings that can be placed on a chessboard so that no two of them are attacking each other?



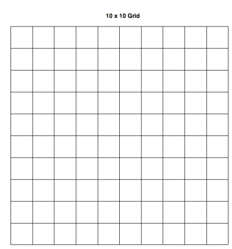
15. Prove that there exist two powers of two which differ by a multiple of 2011.

16. Each box in a 3×3 tic-tac-toe board is filled with one of the numbers -1, 0, 1. Prove that of the eight possible sums along the rows, the columns, and the diagonals, two sums must be equal.



17. Of 40 children seated at a round table, more than half are girls. Prove that there are two girls who are seated diametrically opposite each other.

18. Integers are placed in each square of a 10×10 chessboard, in such a way that no two neighboring integers differ by more than 5. (Two integers are considered neighbors if their squares share a common edge.) Prove that two of the integers must be equal.



19. Prove that among any six people, there are either three people who all know each other or three people who are all strangers to each other. (Assume that if person A knows person B, then person B also knows person A.)

20. Five lattice points are chosen on an infinite square lattice. Prove that the midpoint of one of the segments joining two of these points is also a lattice point.

21. Prove that you can choose a subset of ten given integers such that their sum is divisible by 10.

22. Given 11 different positive integers, none greater than 20, prove that two of these can be chosen, one of which divides the other.